

Bivariate Uniform Deconvolution

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June 9, 2011

Abstract

We construct a density estimator in the bivariate uniform deconvolution model. For this model we derive four inversion formulas to express the bivariate density that we want to estimate in terms of the bivariate density of the observations. By substituting a kernel density estimator of the density of the observations we then get four different estimators. Next we construct an asymptotically optimal convex combination of these four estimators. Expansions for the bias, variance, as well as asymptotic normality, are derived. Some simulated examples are presented.

AMS classification: primary 62G05; secondary 62E20, 62G07, 62G20

Keywords: uniform deconvolution, kernel estimation, bivariate density estimation.

1 Introduction

Before focusing on bivariate deconvolution let us first consider univariate deconvolution. Let X_1, \dots, X_n be i.i.d. observations, where $X_i = Y_i + Z_i$ and Y_i and Z_i are independent. Assume that the unobservable Y_i have distribution function F and density f . Also assume that the unobservable random variables Z_i have a known density k . If the Z_i are uniformly distributed then we have a *uniform deconvolution problem*. Note that the density g of X_i is equal to the convolution of f and k , so $g = k * f$ where $*$ denotes convolution. So we have

$$g(x) = \int_{-\infty}^{\infty} k(x-u)f(u)du. \quad (1)$$

The deconvolution problem is the problem of estimating f or F from the observations X_i .

Several generally applicable methods have been proposed for this deconvolution model. The standard *Fourier type kernel density estimator* for deconvolution problems is based on the Fourier transform, see for instance Wand and Jones (1995). Let w denote a *kernel function* and $h > 0$ a *bandwidth*. The estimator $f_{nh}(x)$ of the density f at the point x is defined as

$$f_{nh}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{\phi_w(ht) \phi_{emp}(t)}{\phi_k(t)} dt = \frac{1}{nh} \sum_{j=1}^n v_h\left(\frac{x - X_j}{h}\right), \quad (2)$$

with

$$v_h(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi_w(s)}{\phi_k(s/h)} e^{-isu} ds, \quad \text{and} \quad \phi_{emp}(t) = \frac{1}{n} \sum_{j=1}^n e^{itX_j},$$

the empirical characteristic function, and ϕ_w and ϕ_k denote the characteristic functions of w and k respectively. An important condition for these estimators to be properly defined is that the characteristic function ϕ_k of the density k has no zeroes, which renders it useless for uniform deconvolution. In fact, Hu and Ridder (2004) argue that in economic applications this assumption is not reasonable since many distributions with a bounded support have characteristic functions with zeros on the real line. They propose an approximation of the Fourier transform estimator in such cases. For other modifications of the Fourier inversion method in this problem see Hall and Meister (2007), Feuerverger, Kim and Sun (2008), Meister (2008) and Delaigle and Meister (2011).

In some univariate deconvolution problems one can apply *nonparametric maximum likelihood*. In the uniform deconvolution problem for instance the error Z is Uniform $[0, 1)$ distributed. So in this particular deconvolution problem we assume to have i.i.d. observations from the density

$$g(x) = \int_{-\infty}^{\infty} I_{[0,1)}(x - u) f(u) du = \int_{x-1}^x f(u) du = F(x) - F(x - 1). \quad (3)$$

Groeneboom and Jongbloed (2003) consider density estimation in this problem. They propose a kernel density estimator based on the nonparametric maximum likelihood estimator (NPMLE) of the distribution function F and derive its asymptotic properties. For estimators of the distribution function in uniform deconvolution, related to the NPMLE, we refer to Groeneboom and Wellner (1992), Van Es and Van Zuijlen (1996) and Donauer, Groeneboom and Jongbloed (2009).

A selected group of deconvolution problems allows explicit *inversion formulas* of (1) expressing the density of interest f in terms of the density g of the data. In these cases we can estimate f by substituting for instance a direct kernel density estimate of g in the inversion formula. In Van Es and Kok (1998) this strategy has been pursued for deconvolution problems where k equals the exponential density, the Laplace density, and their repeated convolutions.

If we apply inversion to the uniform problem then it turns out we get two obvious inversion formulas. Of course these inversions agree on the set of densities of the form (3), but they are different outside of this set. Plugging in a kernel estimator of the density g of the observations, which is typically *not* of this form, then yields two estimators of f . These can then in some sense be optimally combined in a convex combination. This approach is developed in Van Es (2011). Here we will follow this approach in the bivariate uniform deconvolution setting.

Let us now consider *bivariate deconvolution*. The bivariate convolution formula $\mathbf{X}_i = \mathbf{Y}_i + \mathbf{Z}_i$, where \mathbf{X}_i , \mathbf{Y}_i and \mathbf{Z}_i stand for two dimensional random vectors, can be written in vector notation as

$$\begin{pmatrix} X_{i1} \\ X_{i2} \end{pmatrix} = \begin{pmatrix} Y_{i1} \\ Y_{i2} \end{pmatrix} + \begin{pmatrix} Z_{i1} \\ Z_{i2} \end{pmatrix}. \quad (4)$$

The estimation principles described above can in principle all be attempted in the bivariate problem as well. See for instance Youndjé and Wells (2008) for recent results on multivariate Fourier type kernel deconvolution. Approaches based on nonparametric maximum likelihood and inversion hardly exist to our knowledge.

In the bivariate uniform deconvolution setting the random vector \mathbf{Z}_i has a Uniform($[0, 1) \times [0, 1)$) distribution, i.e. it is uniformly distributed on the unit square. Here we can also express the bivariate density g of the observations in terms of the bivariate distribution function F , with density f , of the random vector \mathbf{Y} . We have

$$\begin{aligned} g(x_1, x_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_{[0,1)}(x_1 - u_1) I_{[0,1)}(x_2 - u_2) f(u_1, u_2) du_1 du_2 \\ &= \int_{x_2-1}^{x_2} \int_{x_1-1}^{x_1} f(u_1, u_2) du_1 du_2 \\ &= F(x_1, x_2) - F(x_1, x_2 - 1) - F(x_1 - 1, x_2) + F(x_1 - 1, x_2 - 1). \end{aligned} \quad (5)$$

This is the bivariate analogue of formula (3). Note that, again, the Fourier inversion approach can not be used because of the zeros in the characteristic function of the bivariate uniform distribution.

Apart from being of theoretical interest, bivariate uniform deconvolution is also of interest because of its relation to what one might call *quadrant censoring* or *bivariate current status data*, i.e. a bivariate version of univariate Type I interval censoring. This censoring problem can be described as follows. For convenience we restrict ourselves to the unit square. Consider n i.i.d random points $\mathbf{T}_i, i = 1, \dots, n$, with $\mathbf{T}_i = (T_{i1}, T_{i2})$, in the unit square. Furthermore we have n i.i.d unobservable random points $\mathbf{X}_i, i = 1, \dots, n$, with $\mathbf{X}_i = (X_{i1}, X_{i2})$, also in the unit square. For each i we observe whether \mathbf{X}_i is in quadrant 1, 2, 3 or 4 relative to the known point \mathbf{T}_i . Let us quantify these observations by the discrete random variable Δ_i . So we have

$$\Delta_i = \begin{cases} 1 & , \text{ if } X_{i1} \geq T_{i1} \text{ and } X_{i2} \geq T_{i2}, \\ 2 & , \text{ if } X_{i1} < T_{i1} \text{ and } X_{i2} \geq T_{i2}, \\ 3 & , \text{ if } X_{i1} < T_{i1} \text{ and } X_{i2} < T_{i2}, \\ 4 & , \text{ if } X_{i1} \geq T_{i1} \text{ and } X_{i2} < T_{i2}. \end{cases} \quad (6)$$

This problem is related to uniform deconvolution by a transformation of the data. Assume that the unobserved \mathbf{X}_i have a bivariate density f . The statistical problem is to estimate this density from the observations $(\mathbf{T}_1, \Delta_1), \dots, (\mathbf{T}_n, \Delta_n)$.

Consider the following transformation of the points \mathbf{T}_i ,

$$\mathbf{V}_i = (V_{i1}, V_{i2}) = \begin{cases} (T_{i1} + 1, T_{i2} + 1) & , \text{ if } \Delta_i = 1, \\ (T_{i1}, T_{i2} + 1) & , \text{ if } \Delta_i = 2, \\ (T_{i1}, T_{i2}) & , \text{ if } \Delta_i = 3, \\ (T_{i1} + 1, T_{i2}) & , \text{ if } \Delta_i = 4. \end{cases} \quad (7)$$

It can be shown that if the density f is concentrated on the unit square and if the observation points \mathbf{T}_i are uniformly distributed on the unit square then the density of the random points \mathbf{V}_i is identical to (5). This shows that a method for bivariate uniform deconvolution of the type developed here can also be used in quadrant censoring.

The main aim of this paper is to develop the inversion approach of Van Es (2011) for bivariate uniform deconvolution. In Chapter 2 we derive four inversion formulas for (5). This yields the same number of possible estimators if we plug in a density estimator of the density g of the observations. In Chapter 3 we combine these estimators in a convex combination which is asymptotically optimal in some sense. The weights of this combination turn out to depend on the unknown distribution F . A general theorem for an estimator with estimated weights is given in Chapter 4. We also present specific estimators of these weights. Simulated examples are presented in Chapter 5. Chapter 6 contains the proofs.

2 Inversion formulas

Recall that the density of the \mathbf{Z}_i is equal to $k(z_1, z_2) = I_{[0,1] \times [0,1]}(z_1, z_2) = I_{[0,1]}(z_1)I_{[0,1]}(z_2)$. This yields formula (5) which expresses $g(x_1, x_2)$ in terms of $F(x_1, x_2)$. Lemma 2.1 below demonstrates that the converse is also feasible.

First note that for

$$\begin{aligned} F^{--}(y_1, y_2) &:= \Pr(Y_1 \leq y_1, Y_2 \leq y_2), \\ F^{-+}(y_1, y_2) &:= \Pr(Y_1 \leq y_1, Y_2 > y_2), \\ F^{+-}(y_1, y_2) &:= \Pr(Y_1 > y_1, Y_2 \leq y_2), \\ F^{++}(y_1, y_2) &:= \Pr(Y_1 > y_1, Y_2 > y_2). \end{aligned}$$

the following equalities hold

$$F^{--}(x_1, x_2) = F(x_1, x_2), \tag{8}$$

$$F^{-+}(x_1, x_2) = F_{Y_1}(x_1) - F(x_1, x_2), \tag{9}$$

$$F^{+-}(x_1, x_2) = F_{Y_2}(x_2) - F(x_1, x_2), \tag{10}$$

$$F^{++}(x_1, x_2) = F(x_1, x_2) - F_{Y_1}(x_1) - F_{Y_2}(x_2) + 1. \tag{11}$$

If we know $F(x_1, x_2)$ and if this function is continuously differentiable over x_1 and x_2 , then we know $f(x_1, x_2)$, because $f(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2)$. In fact, combined with the formulas above, and (5), this gives us four different inversion formulas to obtain f and F from g , as is stated in the following Lemma.

Lemma 2.1 *We have*

$$F^{--}(x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g(x_1 - i, x_2 - j), \quad (12)$$

$$F^{-+}(x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} g(x_1 - i, x_2 + j), \quad (13)$$

$$F^{+-}(x_1, x_2) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} g(x_1 + i, x_2 - j), \quad (14)$$

$$F^{++}(x_1, x_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(x_1 + i, x_2 + j). \quad (15)$$

Assume that $\lim_{x_1 \rightarrow \pm\infty} f(x_1, x_2) = 0$ and $\lim_{x_2 \rightarrow \pm\infty} f(x_1, x_2) = 0$. Furthermore, assume that $g(x_1, x_2)$ is twice mixed continuously differentiable over x_1 and x_2 . Then there are four inversion formulas to recover f from g . We have

$$f(x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1 - i, x_2 - j), \quad (16)$$

$$f(x_1, x_2) = - \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1 - i, x_2 + j), \quad (17)$$

$$f(x_1, x_2) = - \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1 + i, x_2 - j), \quad (18)$$

$$f(x_1, x_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1 + i, x_2 + j). \quad (19)$$

To get some more insight in these inversion formulas note that (5) can be interpreted as a probability for \mathbf{Y} (under F). We have

$$g(x_1, x_2) = P_F(\mathbf{Y} \in (x_1 - 1, x_1] \times (x_2 - 1, x_2]).$$

So $g(x_1, x_2)$ is equal to the probability that \mathbf{Y} belongs to a specific square $(x_1 - 1, x_1] \times (x_2 - 1, x_2]$. Adding up over suitable squares we then get the probability that \mathbf{Y} belongs to a specific quadrant with a given vertex. For a formal proof see Chapter 6.

3 Estimation of the density function

In the previous chapter we have derived inversion formulas that express the density f in terms of the density g of the observations. Now we can use an estimator of g , for which we have observations, to estimate f . For an arbitrary density that is not of the form (5), the inversions

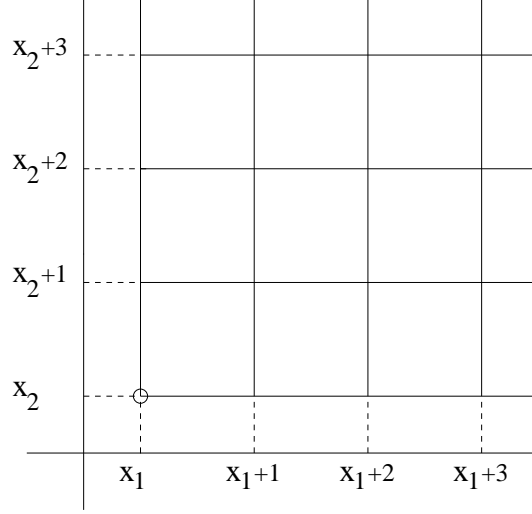


Figure 1: $F^{++}(x_1, x_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_F(\mathbf{Y}_i \in (x_1 + i - 1, x_1 + i] \times (x_2 + j - 1, x_2 + j])$.

will in general not yield distribution functions or densities, nor will they coincide. This typically happens if we estimate g .

We use kernel smoothing but of course other estimators can be used as well. Let us introduce a bivariate kernel density estimator with bivariate kernel function \mathbf{w} and bandwidth $h > 0$. The estimator g_{nh} of g is given by

$$g_{nh}(x_1, x_2) = \frac{1}{nh^2} \sum_{k=1}^n \mathbf{w}\left(\frac{x_1 - X_{k1}}{h}, \frac{x_2 - X_{k2}}{h}\right). \quad (20)$$

Usually, \mathbf{w} is chosen to be a bivariate probability density function. This way it is ensured that g_{nh} is also a density. See for instance Silverman (1986) and Wand and Jones (1995).

We impose the following condition on the kernel function.

Condition W

The function \mathbf{w} is a probability density function on \mathbb{R}^2 with support $[-1, 1] \times [-1, 1]$. Furthermore, we will use a product kernel $\mathbf{w}(u_1, u_2) = w_1(u_1)w_2(u_2)$, where $w_i(u_i)$, with $i \in \{1, 2\}$, denotes a continuously differentiable univariate symmetric probability density function.

We now substitute the kernel estimator in the four inversion formulas of Lemma 2.1. We derive the estimator $f_{nh}^{++}(x_1, x_2)$ as follows. The other three estimators follow similarly. Define

$w'_i(u) := \frac{d}{du}w_i(u)$, $i = 1, 2$. Lemma 2.1 in combination with $\frac{\partial^2}{\partial x_1 \partial x_2}F(x_1, x_2) = f(x_1, x_2)$ gives

$$\begin{aligned} f_{nh}^{++}(x_1, x_2) &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g_{nh}(x_1 + i, x_2 + j) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{\partial^2}{\partial x_1 \partial x_2} \frac{1}{n} \sum_{k=1}^n \frac{1}{h^2} \mathbf{w} \left(\frac{x_1 + i - X_{k1}}{h}, \frac{x_2 + j - X_{k2}}{h} \right) \right) \\ &= \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w'_1 \left(\frac{x_1 + i - X_{k1}}{h} \right) w'_2 \left(\frac{x_2 + j - X_{k2}}{h} \right). \end{aligned}$$

Note that, because of the bounded support of \mathbf{w} , the sum is in fact a finite sum. In the last step we used the fact that \mathbf{w} is a product kernel, and thus $\frac{\partial^2}{\partial u_1 \partial u_2} \mathbf{w}(u_1, u_2) = w'_1(u_1)w'_2(u_2)$.

The four kernel estimators of the density are given by

$$\begin{aligned} f_{nh}^{--}(x_1, x_2) &= \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w'_1 \left(\frac{x_1 - i - X_{k1}}{h} \right) w'_2 \left(\frac{x_2 - j - X_{k2}}{h} \right), \\ f_{nh}^{-+}(x_1, x_2) &= -\frac{1}{nh^4} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} w'_1 \left(\frac{x_1 - i - X_{k1}}{h} \right) w'_2 \left(\frac{x_2 + j - X_{k2}}{h} \right), \\ f_{nh}^{+-}(x_1, x_2) &= -\frac{1}{nh^4} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} w'_1 \left(\frac{x_1 + i - X_{k1}}{h} \right) w'_2 \left(\frac{x_2 - j - X_{k2}}{h} \right), \\ f_{nh}^{++}(x_1, x_2) &= \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w'_1 \left(\frac{x_1 + i - X_{k1}}{h} \right) w'_2 \left(\frac{x_2 + j - X_{k2}}{h} \right). \end{aligned}$$

Next we introduce a convex combination of the four previous estimators. Write

$$f_{nh}^{(t)}(x_1, x_2) = t_1 f_{nh}^{--}(x_1, x_2) + t_2 f_{nh}^{-+}(x_1, x_2) + t_3 f_{nh}^{+-}(x_1, x_2) + t_4 f_{nh}^{++}(x_1, x_2), \quad (21)$$

where $t = (t_1, t_2, t_3, t_4)$ and $t_1 + t_2 + t_3 + t_4 = 1$. For suitable choices of t_1, t_2, t_3, t_4 this combination will turn out to have better properties than any of the estimators separately. Notice that when we set t_1, t_2, t_3 , or t_4 equal to one and the others equal to zero, we get results for $f_{nh}^{--}, f_{nh}^{-+}, f_{nh}^{+-}$, or f_{nh}^{++} individually.

Theorem 3.1 *Assume that Condition W is satisfied, that f is bounded, and that $\lim_{x_1 \rightarrow \pm\infty} f(x_1, x_2) = \lim_{x_2 \rightarrow \pm\infty} f(x_1, x_2) = 0$. If f is twice continuously differentiable on a neighborhood of $x = (x_1, x_2)$ then, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$, we have*

$$\mathbb{E} f_{nh}^{(t)}(x_1, x_2) = f(x_1, x_2) + \frac{1}{2} h^2 \left(\int_{-\infty}^{\infty} z^2 w_1(z) dz f_{11}(x_1, x_2) + \int_{-\infty}^{\infty} z^2 w_2(z) dz f_{22}(x_1, x_2) \right) + o(h^2). \quad (22)$$

Furthermore, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$, we have

$$\text{Var}(f_{nh}^{(t)}(x_1, x_2)) = \frac{1}{nh^6} B(x_1, x_2, t_1, t_2, t_3, t_4) \int_{-1}^1 w'_1(z)^2 dz \int_{-1}^1 w'_2(z)^2 dz + o(n^{-1}h^{-6}) \quad (23)$$

where

$$B(x_1, x_2, t_1, t_2, t_3, t_4) = (t_1^2 F^{--} + t_2^2 F^{-+} + t_3^2 F^{+-} + t_4^2 F^{++})(x_1, x_2). \quad (24)$$

In the proof of the theorem we will see that the expectation of $f_{nh}^{(t)}(x_1, x_2)$ is the same whatever convex combination we choose for. Lemma 3.2 gives the weights that minimize the leading term in the variance (23).

Lemma 3.2 *Assume that (x_1, x_2) is an interior point of the support of f . The weights t_1, t_2, t_3 and t_4 , with $t_1 + t_2 + t_3 + t_4 = 1$, that minimize the leading term in the variance (23), are denoted by $\bar{t}_1(x_1, x_2), \bar{t}_2(x_1, x_2), \bar{t}_3(x_1, x_2)$ and $\bar{t}_4(x_1, x_2)$ and they are equal to*

$$\begin{aligned}\bar{t}_1(x_1, x_2) &= F^{-+,+-,++}(x_1, x_2)A(x_1, x_2), \\ \bar{t}_2(x_1, x_2) &= F^{--,+-,++}(x_1, x_2)A(x_1, x_2), \\ \bar{t}_3(x_1, x_2) &= F^{-,-,+ ,++}(x_1, x_2)A(x_1, x_2), \\ \bar{t}_4(x_1, x_2) &= F^{-,-,+ ,+-}(x_1, x_2)A(x_1, x_2).\end{aligned}$$

The resulting variance of this optimal convex combination is then equal to

$$\text{Var}(f_{nh}(x_1, x_2)) = A(x_1, x_2)C(x_1, x_2)\frac{1}{nh^6}\int_{-1}^1 w'_1(z)^2 dz \int_{-1}^1 w'_2(z)^2 dz + o(n^{-1}h^{-6}), \quad (25)$$

Here

$$A(x_1, x_2) := (F^{-+,+-,++} + F^{--,+-,++} + F^{-,-,+ ,++} + F^{-,-,+ ,+-})^{-1}(x_1, x_2). \quad (26)$$

where, for $a_1, a_2, b_1, b_2, c_1, c_2 \in \{-, +\}$,

$$F^{a_1 a_2, b_1 b_2, c_1 c_2}(x_1, x_2) := F^{a_1 a_2}(x_1, x_2)F^{b_1 b_2}(x_1, x_2)F^{c_1 c_2}(x_1, x_2), \quad (27)$$

and

$$C(x_1, x_2) := F^{--}(x_1, x_2)F^{-+}(x_1, x_2)F^{+-}(x_1, x_2)F^{++}(x_1, x_2). \quad (28)$$

Proof

First note that the weights are well defined since the fact that (x_1, x_2) is an interior point of the support of f implies that $F^{--}(x_1, x_2), F^{-+}(x_1, x_2), F^{+-}(x_1, x_2)$ and $F^{++}(x_1, x_2)$ are strictly positive. The lower bound now follows from Lemma 6.2 in Chapter 6. \square

Note that in general, of course, we do not know F . However, in Section 4 we show that we can estimate $F^{--}(x_1, x_2), F^{-+}(x_1, x_2), F^{+-}(x_1, x_2)$, and $F^{++}(x_1, x_2)$, again using the inversion formulas of Theorem 2.1. This will lead to estimates of the optimal weights. We then prove that the estimator with estimated weights shares the properties of Theorem 3.1 with the optimal weights.

4 The final estimator with estimated optimal weights

Let us write $\hat{t}_n(x_1, x_2) = (\hat{t}_{n1}(x_1, x_2), \dots, \hat{t}_{n4}(x_1, x_2))$ for a vector of estimated weights. The next theorem shows that under some conditions on these estimators the limit behaviour of $f_{nh}^{(\hat{t}_n)}(x_1, x_2)$ resembles the optimal limit behaviour of the estimator $f_{nh}^{(\bar{t})}(x_1, x_2)$.

Theorem 4.1 Assume that Condition W is satisfied, that f is bounded, and that $\lim_{x_1 \rightarrow \pm\infty} f(x_1, x_2) = \lim_{x_1 \rightarrow \pm\infty} f(x_1, x_2) = 0$.

Assume for $i = 1, \dots, 4$,

$$E(\hat{t}_{ni}(x_1, x_2) - \bar{t}_i(x_1, x_2))^2 = o(nh^{10}). \quad (29)$$

If f is twice continuously differentiable on a neighborhood of $x = (x_1, x_2)$ then, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$, we have

$$E f_{nh}^{(\hat{t}_n)}(x_1, x_2) = f(x_1, x_2) + \frac{1}{2}h^2 \left(\int_{-\infty}^{\infty} z^2 w_1(z) dz f_{11}(x_1, x_2) + \int_{-\infty}^{\infty} z^2 w_2(z) dz f_{22}(x_1, x_2) \right) + o(h^2). \quad (30)$$

Assume for $i = 1, \dots, 4$,

$$E(\hat{t}_{ni}(x_1, x_2) - \bar{t}_i(x_1, x_2))^4 = o(1). \quad (31)$$

Then, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$, we have

$$\text{Var}(f_{nh}^{(\hat{t}_n)}(x_1, x_2)) = \frac{1}{nh^6} \sigma(x_1, x_2)^2 + o(n^{-1}h^{-6}), \quad (32)$$

where, with the notation of Lemma 3.2, $\sigma(x_1, x_2)^2$ is defined by

$$\sigma(x_1, x_2)^2 = A(x_1, x_2)C(x_1, x_2) \int_{-1}^1 w_1'(z_1)^2 dz_1 \int_{-1}^1 w_2'(z_2)^2 dz_2. \quad (33)$$

Assume for $i = 1, \dots, 4$,

$$E(\hat{t}_{ni}(x_1, x_2) - \bar{t}_i(x_1, x_2))^2 = o(1). \quad (34)$$

Then the estimator is asymptotically normally distributed. We have, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$,

$$\sqrt{nh}^3 \left(f_{nh}^{(\hat{t}_n)}(x_1, x_2) - E f_{nh}^{(\hat{t}_n)}(x_1, x_2) \right) \xrightarrow{D} N(0, \sigma(x_1, x_2)^2). \quad (35)$$

Let us next construct suitable estimators of the weights based on the estimators of F^{--}, F^{-+}, F^{+-} and F^{++} . As in estimation of the density we can plug in (20) into the inversion formulas for F in Lemma 2.1 and get kernel estimators of $F^{--}(x_1, x_2), F^{-+}(x_1, x_2), F^{+-}(x_1, x_2)$ and $F^{++}(x_1, x_2)$. We get four estimators, given by

$$\begin{aligned} F_{nh}^{--}(x_1, x_2) &= \frac{1}{nh^2} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_1 \left(\frac{x_1 - i - X_{k1}}{h} \right) w_2 \left(\frac{x_2 - j - X_{k2}}{h} \right), \\ F_{nh}^{-+}(x_1, x_2) &= \frac{1}{nh^2} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} w_1 \left(\frac{x_1 - i - X_{k1}}{h} \right) w_2 \left(\frac{x_2 + j - X_{k2}}{h} \right), \\ F_{nh}^{+-}(x_1, x_2) &= \frac{1}{nh^2} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} w_1 \left(\frac{x_1 + i - X_{k1}}{h} \right) w_2 \left(\frac{x_2 - j - X_{k2}}{h} \right), \\ F_{nh}^{++}(x_1, x_2) &= \frac{1}{nh^2} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w_1 \left(\frac{x_1 + i - X_{k1}}{h} \right) w_2 \left(\frac{x_2 + j - X_{k2}}{h} \right). \end{aligned} \quad (36)$$

The following theorem establishes the asymptotic bias and variance of these four estimators. In the sequel we adopt the notation $F_{11}^{--} = \frac{\partial^2 F^{--}(x_1, x_2)}{\partial x_1^2}$ and $F_{12}^{--} = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2}$, etc., also for the density f . The proof is very similar to the proof of Theorem 3.1 and is therefore omitted. See Benešová et al. (2011) for a complete proof.

Theorem 4.2 *Assume that Condition W is satisfied. Then, as $n \rightarrow \infty$, $h \rightarrow 0$, $nh \rightarrow \infty$ we have*

$$\begin{aligned} E F_{nh}^{--}(x_1, x_2) &= F^{--}(x_1, x_2) + \frac{1}{2}h^2 \left(\int_{-\infty}^{\infty} z^2 w_1(z) dz F_{11}^{--}(x_1, x_2) + \int_{-\infty}^{\infty} z^2 w_2(z) dz F_{22}^{--}(x_1, x_2) \right) + o(h^2), \\ E F_{nh}^{-+}(x_1, x_2) &= F^{-+}(x_1, x_2) + \frac{1}{2}h^2 \left(\int_{-\infty}^{\infty} z^2 w_1(z) dz F_{11}^{-+}(x_1, x_2) + \int_{-\infty}^{\infty} z^2 w_2(z) dz F_{22}^{-+}(x_1, x_2) \right) + o(h^2), \\ E F_{nh}^{+-}(x_1, x_2) &= F^{+-}(x_1, x_2) + \frac{1}{2}h^2 \left(\int_{-\infty}^{\infty} z^2 w_1(z) dz F_{11}^{+-}(x_1, x_2) + \int_{-\infty}^{\infty} z^2 w_2(z) dz F_{22}^{+-}(x_1, x_2) \right) + o(h^2), \\ E F_{nh}^{++}(x_1, x_2) &= F^{++}(x_1, x_2) + \frac{1}{2}h^2 \left(\int_{-\infty}^{\infty} z^2 w_1(z) dz F_{11}^{++}(x_1, x_2) + \int_{-\infty}^{\infty} z^2 w_2(z) dz F_{22}^{++}(x_1, x_2) \right) + o(h^2). \end{aligned}$$

For the variances we have

$$\begin{aligned} \text{Var}(F_{nh}^{--}(x_1, x_2)) &= F^{--}(x_1, x_2) \frac{1}{nh^2} \int_{-1}^1 w_1^2(z) dz \int_{-1}^1 w_2^2(z) dz + o\left(\frac{1}{nh^2}\right), \\ \text{Var}(F_{nh}^{-+}(x_1, x_2)) &= F^{-+}(x_1, x_2) \frac{1}{nh^2} \int_{-1}^1 w_1^2(z) dz \int_{-1}^1 w_2^2(z) dz + o\left(\frac{1}{nh^2}\right), \\ \text{Var}(F_{nh}^{+-}(x_1, x_2)) &= F^{+-}(x_1, x_2) \frac{1}{nh^2} \int_{-1}^1 w_1^2(z) dz \int_{-1}^1 w_2^2(z) dz + o\left(\frac{1}{nh^2}\right), \\ \text{Var}(F_{nh}^{++}(x_1, x_2)) &= F^{++}(x_1, x_2) \frac{1}{nh^2} \int_{-1}^1 w_1^2(z) dz \int_{-1}^1 w_2^2(z) dz + o\left(\frac{1}{nh^2}\right). \end{aligned}$$

For the proof of this theorem see Chapter 6.

Next we write the optimal weights of Lemma 3.2 in terms of functions \tilde{t}_i defined by

$$\bar{t}_i(x_1, x_2) = \tilde{t}_i(F^{--}(x_1, x_2), F^{-+}(x_1, x_2), F^{+-}(x_1, x_2), F^{++}(x_1, x_2)), \quad i = 1, \dots, 4.$$

Let (ϵ_n) denote a sequence of numbers with $0 < \epsilon_n < 1$ and $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Then define truncated versions of the estimators $F_{nh}^{--}(x_1, x_2)$, $F_{nh}^{-+}(x_1, x_2)$, $F_{nh}^{+-}(x_1, x_2)$, $F_{nh}^{++}(x_1, x_2)$ and $F_{nh}^{++}(x_1, x_2)$ by

$$\begin{aligned} \tilde{F}_{nh}^{--}(x_1, x_2) &= \min(\max(F_{nh}^{--}(x_1, x_2), \epsilon_n), 1), \\ \tilde{F}_{nh}^{-+}(x_1, x_2) &= \min(\max(F_{nh}^{-+}(x_1, x_2), \epsilon_n), 1), \\ \tilde{F}_{nh}^{+-}(x_1, x_2) &= \min(\max(F_{nh}^{+-}(x_1, x_2), \epsilon_n), 1), \\ \tilde{F}_{nh}^{++}(x_1, x_2) &= \min(\max(F_{nh}^{++}(x_1, x_2), \epsilon_n), 1). \end{aligned}$$

Since the bandwidth used in the estimators of the weights can in general be different to the bandwidth h used in the estimator of f , we will denote this bandwidth by \tilde{h} . We now obtain estimators of the weights by plugging in these estimators. We get

$$\hat{t}_{ni}(x_1, x_2) = \tilde{t}_i(\tilde{F}_{n\tilde{h}}^{--}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{-+}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{+-}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{++}(x_1, x_2)), \quad i = 1, \dots, 4.$$

The next lemma shows that these estimators, with a suitable bandwidth, can be used to estimate the optimal weights without disturbing the asymptotics of Theorem 3.1.

Lemma 4.3 *If $h \gg n^{-1/6}$, $\epsilon_n = 1/\log n$, and if we use a bandwidth \tilde{h} of the form $\tilde{h} = cn^{-1/6}$, where c is a constant, then the estimators*

$$\hat{t}_{ni}(x_1, x_2) = \tilde{t}_i(\tilde{F}_{n\tilde{h}}^{--}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{-+}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{+-}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{++}(x_1, x_2))$$

satisfy (29), (31) and (34).

Remark 4.4 *If we compare the performance of our final estimator with estimated optimal weights to the performance of the four individual estimators then we see that the first order of the expectation is the same. The variance of the combined estimator contains the term $C(x_1, x_2)$ which is equal to the product of $F^{--}(x_1, x_2)$, $F^{-+}(x_1, x_2)$, $F^{+-}(x_1, x_2)$ and $F^{++}(x_1, x_2)$. This shows that the variance is small along the edge of the support of f . By Theorem 3.1 the variance of, for instance, $\tilde{f}_{n\tilde{h}}^{--}(x_1, x_2)$ is proportional to $F^{--}(x_1, x_2)$. So this estimator will perform better in the lower left of the support of f than it will in the other part. By using the estimated optimal convex combination the worse behavior of the four individual estimators in certain areas is reduced.*

Remark 4.5 *Since in the theorems we use a bivariate kernel function \mathbf{w} which is the product of two different univariate density functions w_1 and w_2 , in fact we allow different bandwidths for the two coordinates, provided the bandwidths are of the same order. Writing $h_1 = h$, $h_2 = ch$, $w_1 = w$ and $w_2 = w(\cdot/c)/c$, for some $c > 0$, and writing $f_{nh_1h_2}^{(t)}$ for the resulting estimator, we get the following leading terms in the expansions of its bias and variance in Theorem 3.1,*

$$\frac{1}{2} \int_{-\infty}^{\infty} z^2 w(z) dz \left(h_1^2 f_{11}(x_1, x_2) + h_2^2 f_{22}(x_1, x_2) \right) \quad (37)$$

and

$$\frac{1}{nh_1^3 h_2^3} B(x_1, x_2, t_1, t_2, t_3, t_4) \left(\int_{-1}^1 w'(z)^2 dz \right)^2. \quad (38)$$

The subsequent theorems can be likewise adapted to different bandwidths.

Remark 4.6 *If we minimize the pointwise asymptotic mean squared error of $f_{n\tilde{h}}^{(\hat{t}_n)}(x_1, x_2)$ and thus balance its asymptotic squared bias and its asymptotic variance given by Theorem 4.1 then we see that the optimal bandwidth is of order $n^{-1/10}$. The corresponding mean squared error is then equal to $n^{-2/5}$. This of course raises the problem of bandwidth selection which, important though as it is for applications, we will not pursue here.*

Remark 4.7 *In the proofs we see that the bias of our final estimator is asymptotically of the same form as the bias of a bivariate kernel density estimator based on direct observations. That means that, if the smoothness assumptions on the density f are strengthened, bias reduction techniques, such as for instance higher order kernels or even super kernels, can be used to increase the rate of convergence.*

Remark 4.8 *The construction as presented here for bivariate data can in principle also be done for arbitrary dimension d . For dimension one we have to combine two inversion formulas as shown in Van Es (2011). In the present paper, for dimension two, we combine four inversion formulas, and for arbitrary dimension d combination of 2^d inversions has to be accomplished. Of course the complexity of the estimator will increase rapidly with growing dimension.*

5 Simulated examples

To illustrate the estimator we have simulated two examples. In the first example the density f is unimodal. In the second example f is a mixture of two unimodal bivariate densities, rendering it bimodal. In the first example f is concentrated on the square $[0.25, 1.75] \times [0.25, 1.75]$. In the second example f is concentrated on the square $[0.2, 1.8] \times [0.2, 1.8]$. This means that both deconvolution problems are not at all trivial.

To speed up computations we have followed the bivariate binning technique as advised in Wand (1994). For the x and y coordinates we have chosen for a grid of 500 points between -1 and 4. We have used a product kernel based on the so called biweight kernel given by

$$w_1(u) = w_2(u) = \frac{15}{16} (1 - u^2)^2 I_{[-1,1]}(u). \quad (39)$$

Example 5.1 In our first example f is the density of the random vector (Y_1, Y_2) , where Y_1 and Y_2 are two independent random variables that each have a certain shifted and rescaled beta distribution. To be more specific $Y_i = 0.25 + 1.5V_i, i = 1, 2$, where the V_i are independent and both Beta(3,3) distributed. We have simulated 1000 values so $n = 1000$. The bandwidth h , chosen by hand, is equal to 0.5.

The true density f and its estimate are given in Figure 2. The difference between the true density and the estimate is plotted in Figure 3. The right plot in Figure 3 shows f_{nh}^{+-} . Clearly this estimate is best in the $+-$ quadrant, as predicted by the theory.

Example 5.2 In our second example f is the density of the random vector (Y_1, Y_2) , where Y_1 and Y_2 are dependent random variables with a bimodal distribution. The distribution of the vector is a mixture of two distributions like the one in Example 5.1. The values of the Y 's are generated as follows. With V_1 and V_2 having the same distribution as in the previous example the Y values are given by

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} V_1 + 0.2 \\ V_2 + 0.8 \end{pmatrix} & , \text{ with probability } 2/5, \\ \begin{pmatrix} V_1 + 0.8 \\ V_2 + 0.2 \end{pmatrix} & , \text{ with probability } 3/5. \end{cases}$$

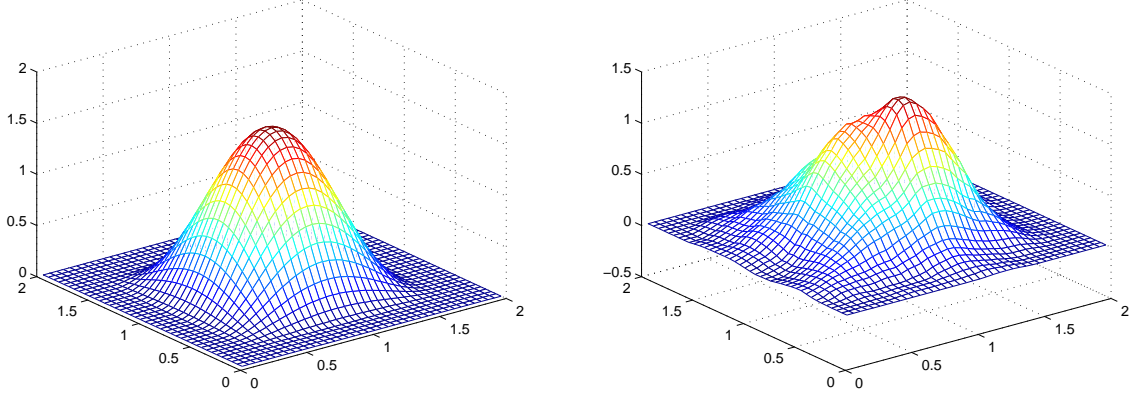


Figure 2: Left: the true density. Right: the estimate.

We have simulated 5000 values so $n = 5000$. The bandwidth h , chosen by hand, is equal to 0.35.

The true density f and its estimate are given in Figure 4. The difference between the true density and the estimate is plotted in Figure 5. The right plot in Figure 5 shows f_{nh}^{-+} . Clearly this estimate is best in the $-+$ quadrant, as predicted by the theory.

6 Proofs

6.1 Proof of Lemma 2.1

Let us first derive the inversion formulas for $F(x_1, x_2)$. We sum $g(x_1 - i, x_2) = F(x_1 - i, x_2) - F(x_1 - i, x_2 - 1) - F(x_1 - i - 1, x_2) + F(x_1 - i - 1, x_2 - 1)$ over the first coordinate to obtain two telescopic sums. Thus we get

$$\begin{aligned}
& \sum_{i=0}^{\infty} g(x_1 - i, x_2) \\
&= \sum_{i=0}^{\infty} \{F(x_1 - i, x_2) - F(x_1 - i, x_2 - 1) - F(x_1 - i - 1, x_2) + F(x_1 - i - 1, x_2 - 1)\} \\
&= \sum_{i=0}^{\infty} \{F(x_1 - i, x_2) - F(x_1 - i - 1, x_2)\} - \sum_{i=0}^{\infty} \{F(x_1 - i, x_2 - 1) - F(x_1 - i - 1, x_2 - 1)\} \\
&= F(x_1, x_2) - F(x_1, x_2 - 1).
\end{aligned} \tag{40}$$

Here we used that $\lim_{i \rightarrow \infty} F(x_1 - i, x_2) = \lim_{i \rightarrow \infty} F(x_1 - i, x_2 - 1) = 0$, for F is a bivariate distribution function. Next, we sum over the second coordinate. Because we also have

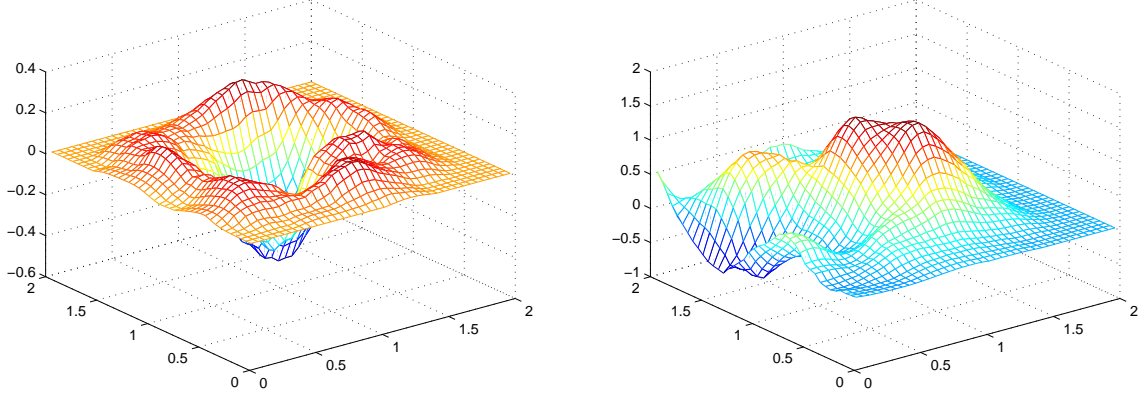


Figure 3: Left: the difference of the true density and the estimate. Right: f_{nh}^{+-} .

$\lim_{j \rightarrow \infty} F(x_1, x_2 - j) = 0$, we get

$$\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} g(x_1 - i, x_2 - j) = \sum_{j=0}^{\infty} \{F(x_1, x_2 - j) - F(x_1, x_2 - j - 1)\} = F(x_1, x_2). \quad (41)$$

Because the terms are nonnegative, the order of summation can be interchanged and we have shown (12). Thus we have found an expression for the unobservable probability distribution function F in terms of the observable density function g .

Above, we iterated over $-i$, so now let us determine what happens when we iterate over $+i$. First, we write $g(x_1 + i, x_2)$ as

$$g(x_1 + i, x_2) = F(x_1 + i, x_2) - F(x_1 + i, x_2 - 1) - F(x_1 + i - 1, x_2) + F(x_1 + i - 1, x_2 - 1). \quad (42)$$

Secondly, we take the sum over the first coordinate. Again we get two telescopic sums. Note that $\lim_{i \rightarrow \infty} F(x_1 + i, x_2) = F_{Y_2}(x_2)$ and $\lim_{i \rightarrow \infty} F(x_1 + i, x_2 - 1) = F_{Y_2}(x_2 - 1)$, so we get

$$\begin{aligned} & \sum_{i=1}^{\infty} g(x_1 + i, x_2) \\ &= \sum_{i=1}^{\infty} \{F(x_1 + i, x_2) - F(x_1 + i, x_2 - 1) - F(x_1 + i - 1, x_2) + F(x_1 + i - 1, x_2 - 1)\} \\ &= \sum_{i=1}^{\infty} \{F(x_1 + i, x_2) - F(x_1 + i - 1, x_2)\} + \sum_{i=1}^{\infty} \{F(x_1 + i - 1, x_2 - 1) - F(x_1 + i, x_2 - 1)\} \\ &= F_{Y_2}(x_2) - F(x_1, x_2) + F(x_1, x_2 - 1) - F_{Y_2}(x_2 - 1). \end{aligned} \quad (43)$$

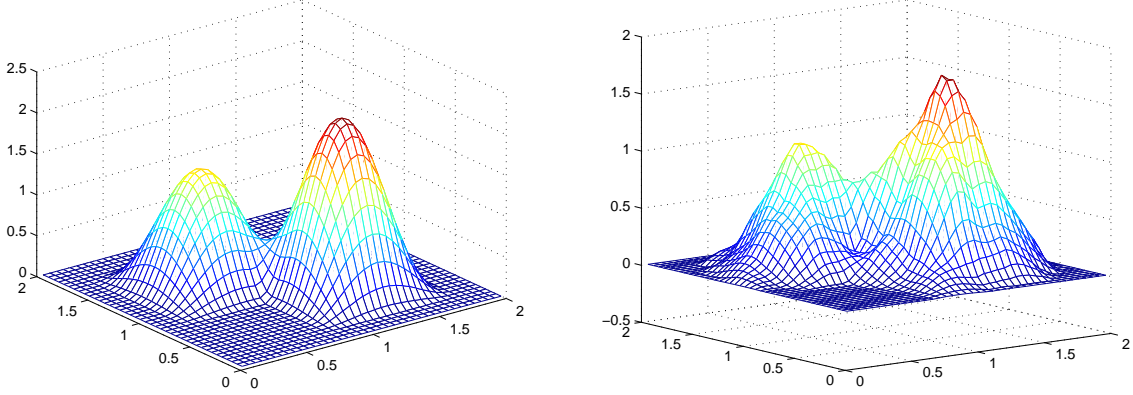


Figure 4: $n = 5000, h = 0.35$. Left: the true density. Right: the estimate.

Thirdly, we sum over the second coordinate. Because $\lim_{j \rightarrow \infty} F_{Y_2}(x_2 - j) = 0$, this results in

$$\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} g(x_1 + i, x_2 - j) \\
&= \sum_{j=0}^{\infty} \{F_{Y_2}(x_2 - j) - F(x_1, x_2 - j) + F(x_1, x_2 - j - 1) - F_{Y_2}(x_2 - j - 1)\} \\
&= \sum_{j=0}^{\infty} \{F_{Y_2}(x_2 - j) - F_{Y_2}(x_2 - j - 1)\} - \sum_{j=0}^{\infty} \{F(x_1, x_2 - j) - F(x_1, x_2 - j - 1)\} \\
&= F_{Y_2}(x_2) - F(x_1, x_2) = F^{+-}(x_1, x_2).
\end{aligned} \tag{44}$$

Again, we can interchange the sums and we have shown (14). In similar fashion we can derive (13).

The last formula to recover $F(x_1, x_2)$ can be derived as follows. We begin with

$$g(x_1 + 1, x_2 + 1) = F(x_1, x_2) - F(x_1, x_2 + 1) - F(x_1 + 1, x_2) + F(x_1 + 1, x_2 + 1). \tag{45}$$

Now sum over the first coordinate to obtain

$$\sum_{i=1}^{\infty} g(x_1 + i, x_2 + 1) = F(x_1, x_2) - F(x_1, x_2 + 1) - F_{Y_2}(x_2) + F_{Y_2}(x_2 + 1). \tag{46}$$

Summing over the second coordinate we get

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} g(x_1 + i, x_2 + j) = F(x_1, x_2) - F_{Y_1}(x_1) - F_{Y_2}(x_2) + 1 = F^{++}(x_1, x_2). \tag{47}$$

Changing the order of summation again, we obtain (15).

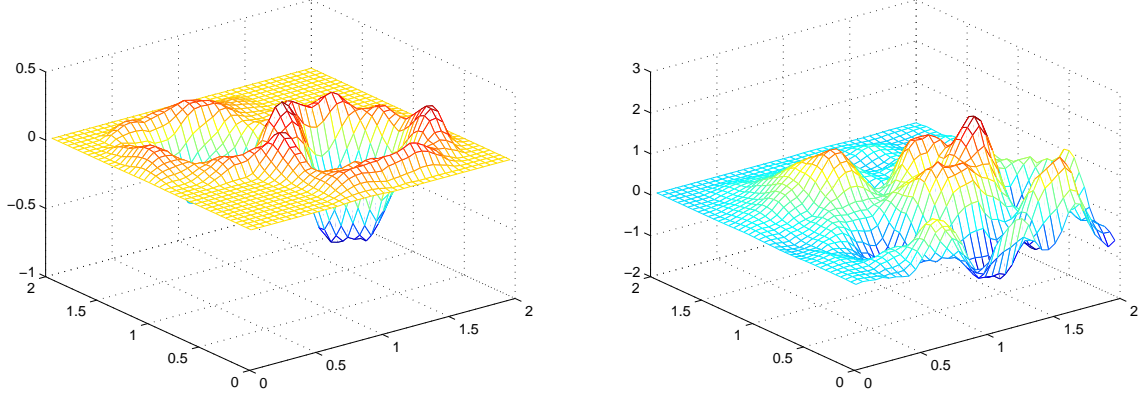


Figure 5: Left: the difference of the true density and the estimate. Right: f_{nh}^{-+} .

The four inversion formulas for f are derived in a similar fashion. From (5) we have

$$\frac{\partial^2}{\partial x_1 \partial x_2} g(x_1, x_2) = f(x_1, x_2) - f(x_1, x_2 - 1) - f(x_1 - 1, x_2) + f(x_1 - 1, x_2 - 1).$$

Now, following equations (40) and (41), we obtain

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1 - i, x_2 - j) = f(x_1, x_2). \quad (48)$$

Here we have used $\lim_{x_1 \rightarrow -\infty} f(x_1, x_2) = 0$ and $\lim_{x_2 \rightarrow -\infty} f(x_1, x_2) = 0$.

The other three inversion formulas follow similarly. □

6.2 Proof of Theorem 3.1

First we consider the estimator f_{nh}^{++} . We have

$$\begin{aligned} \mathbb{E} f_{nh}^{++}(x_1, x_2) &= \mathbb{E} \left(\frac{1}{nh^4} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w_1' \left(\frac{x_1 + i - X_{k1}}{h} \right) w_2' \left(\frac{x_2 + j - X_{k2}}{h} \right) \right) \\ &= \frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{E} w_1' \left(\frac{x_1 + i - X_{11}}{h} \right) w_2' \left(\frac{x_2 + j - X_{12}}{h} \right) \\ &= \frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1' \left(\frac{x_1 + i - u_1}{h} \right) w_2' \left(\frac{x_2 + j - u_2}{h} \right) g(u_1, u_2) du_1 du_2. \end{aligned} \quad (49)$$

Note that interchanging integrals and sums is allowed because

$$\frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| w_1' \left(\frac{x_1 + i - u_1}{h} \right) \right| \left| w_2' \left(\frac{x_2 + j - u_2}{h} \right) \right| g(u_1, u_2) du_1 du_2 < \infty. \quad (50)$$

To check this, we first make the substitutions $v_1 := u_1 - i$ and $v_2 := u_2 - j$. Secondly, we interchange the sums and integrals again, which is allowed because the integrand is nonnegative (Fubini). We get

$$\begin{aligned} & \frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| w'_1 \left(\frac{x_1 - v_1}{h} \right) \right| \left| w'_2 \left(\frac{x_2 - v_2}{h} \right) \right| g(v_1 + i, v_2 + j) dv_1 dv_2 \\ &= \frac{1}{h^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| w'_1 \left(\frac{x_1 - v_1}{h} \right) \right| \left| w'_2 \left(\frac{x_2 - v_2}{h} \right) \right| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(v_1 + i, v_2 + j) dv_1 dv_2. \end{aligned} \quad (51)$$

Thirdly, noting that $F^{++}(v_1, v_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(v_1 + i, v_2 + j) dv_1 dv_2$ and that $F^{++}(v_1, v_2) \leq 1$, we obtain

$$\begin{aligned} & \frac{1}{h^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| w'_1 \left(\frac{x_1 - v_1}{h} \right) \right| \left| w'_2 \left(\frac{x_2 - v_2}{h} \right) \right| F^{++}(v_1, v_2) dv_1 dv_2 \\ & \leq \frac{1}{h^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| w'_1 \left(\frac{x_1 - v_1}{h} \right) \right| \left| w'_2 \left(\frac{x_2 - v_2}{h} \right) \right| dv_1 dv_2 < \infty. \end{aligned} \quad (52)$$

Because w'_1 and w'_2 are bounded functions, and have bounded support, this integral is finite. Thus our use of Fubini's Theorem is justified. Next we apply partial integration twice, yielding

$$\begin{aligned} \mathbb{E} f_{nh}^{++}(x_1, x_2) &= \frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} w'_2 \left(\frac{x_2 + j - u_2}{h} \right) \left(\int_{-\infty}^{\infty} w'_1 \left(\frac{x_1 + i - u_1}{h} \right) g(u_1, u_2) du_1 \right) du_2 \\ &= -\frac{1}{h^3} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} w'_2 \left(\frac{x_2 + j - u_2}{h} \right) \left(\int_{-\infty}^{\infty} w_1 \left(\frac{x_1 + i - u_1}{h} \right) \frac{\partial}{\partial u_1} g(u_1, u_2) du_1 \right) du_2 \\ &= -\frac{1}{h^3} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} w_1 \left(\frac{x_1 + i - u_1}{h} \right) \left(\int_{-\infty}^{\infty} w'_2 \left(\frac{x_2 + j - u_2}{h} \right) \frac{\partial}{\partial u_1} g(u_1, u_2) du_2 \right) du_1 \\ &= \frac{1}{h^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 \left(\frac{x_1 + i - u_1}{h} \right) w_2 \left(\frac{x_2 + j - u_2}{h} \right) \frac{\partial^2}{\partial u_1 \partial u_2} g(u_1, u_2) du_1 du_2. \end{aligned}$$

By the substitutions $v_1 := u_1 - i$ and $v_2 := u_2 - j$ we get

$$\begin{aligned} & \mathbb{E} f_{nh}^{++}(x_1, x_2) \\ &= \frac{1}{h^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 \left(\frac{x_1 - v_1}{h} \right) w_2 \left(\frac{x_2 - v_2}{h} \right) \frac{\partial^2}{\partial u_1 \partial u_2} g(v_1 + i, v_2 + j) dv_1 dv_2. \end{aligned} \quad (53)$$

Now we need to interchange integrals and sums again. Therefore, rewrite the equation above as

$$\begin{aligned}
& \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1\left(\frac{x_1 - v_1}{h}\right) w_2\left(\frac{x_2 - v_2}{h}\right) \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) dv_1 dv_2 \\
&= \lim_{M_1 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1\left(\frac{x_1 - v_1}{h}\right) w_2\left(\frac{x_2 - v_2}{h}\right) \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) dv_1 dv_2 \\
&= \lim_{M_1 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1\left(\frac{x_1 - v_1}{h}\right) w_2\left(\frac{x_2 - v_2}{h}\right) \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) dv_1 dv_2.
\end{aligned} \tag{54}$$

By (42) we have $g(v_1 + i, v_2) = F(v_1 + i, v_2) - F(v_1 + i, v_2 - 1) - F(v_1 + i - 1, v_2) + F(v_1 + i - 1, v_2 - 1)$, so

$$\frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2) = f(v_1 + i, v_2) - f(v_1 + i, v_2 - 1) - f(v_1 + i - 1, v_2) + f(v_1 + i - 1, v_2 - 1).$$

Following the summation of (43), we find

$$\begin{aligned}
& \sum_{i=1}^{M_1} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2) \\
&= \sum_{i=1}^{M_1} \{f(v_1 + i, v_2) - f(v_1 + i, v_2 - 1) - f(v_1 + i - 1, v_2) + f(v_1 + i - 1, v_2 - 1)\} \\
&= \sum_{i=1}^{M_1} \{f(v_1 + i, v_2) - f(v_1 + i - 1, v_2)\} + \sum_{i=1}^{M_1} \{f(v_1 + i - 1, v_2 - 1) - f(v_1 + i, v_2 - 1)\} \\
&= f(v_1 + M_1, v_2) - f(v_1, v_2) - f(v_1, v_2 - 1) - f(v_1 + M_1, v_2 - 1)
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
& \sum_{j=1}^{M_2} \sum_{i=1}^{M_1} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) \\
&= \sum_{j=1}^{M_2} \{f(v_1 + M_1, v_2 + j) - f(v_1, v_2 + j) - f(v_1, v_2 + j - 1) - f(v_1 + M_1, v_2 + j - 1)\} \\
&= \sum_{j=1}^{M_2} \{f(v_1 + M_1, v_2 + j) - f(v_1 + M_1, v_2 + j - 1) + \sum_{j=1}^{M_2} \{f(v_1, v_2 + j - 1) - f(v_1, v_2 + j)\}\} \\
&= f(v_1 + M_1, v_2 + M_2) - f(v_1 + M_1, v_2) + f(v_1, v_2) - f(v_1, v_2 + M_2).
\end{aligned} \tag{56}$$

Note that this sum is finite for all v_1, v_2 , because f is bounded. Also note that changing the order of summation is allowed, because $M_1, M_2 < \infty$. By Lemma 2.1 we have

$$\lim_{M_1 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) = f(v_1, v_2) < \infty. \tag{57}$$

We have assumed that f is bounded, so let $f(v_1, v_2) \leq \frac{1}{4}A$ for all v_1, v_2 , where $A > 0$ is a constant. Observe the following inequality

$$\begin{aligned} & |f(v_1 + M_1, v_2 + M_2) - f(v_1 + M_1, v_2) + f(v_1, v_2) - f(v_1, v_2 + M_2)| \\ & \leq |f(v_1 + M_1, v_2 + M_2)| + |f(v_1 + M_1, v_2)| + |f(v_1, v_2)| + |f(v_1, v_2 + M_2)| \\ & \leq A, \end{aligned} \tag{58}$$

for all v_1, v_2, M_1 , and M_2 . Note that, because w_1 and w_2 are nonnegative, bounded and have bounded support,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1\left(\frac{x_1 - v_1}{h}\right) w_2\left(\frac{x_2 - v_2}{h}\right) dv_1 dv_2 < \infty \tag{59}$$

for all x_1, x_2 . Thus we can apply the Lebesgue Dominated Convergence Theorem to (53), and find

$$\begin{aligned} & \lim_{M_1 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1\left(\frac{x_1 - v_1}{h}\right) w_2\left(\frac{x_2 - v_2}{h}\right) \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) dv_1 dv_2 \\ & = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1\left(\frac{x_1 - v_1}{h}\right) w_2\left(\frac{x_2 - v_2}{h}\right) \lim_{M_1 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) dv_1 dv_2 \\ & = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1\left(\frac{x_1 - v_1}{h}\right) w_2\left(\frac{x_2 - v_2}{h}\right) f(v_1, v_2) dv_1 dv_2. \end{aligned} \tag{60}$$

Summarizing we now have

$$\mathbb{E} f_{nh}^{++}(x_1, x_2) = \frac{1}{h^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1\left(\frac{x_1 - v_1}{h}\right) w_2\left(\frac{x_2 - v_2}{h}\right) f(v_1, v_2) dv_1 dv_2. \tag{61}$$

Substituting $z_1 := \frac{x_1 - v_1}{h}$ and $z_2 := \frac{x_2 - v_2}{h}$ we get

$$\mathbb{E} f_{nh}^{++}(x_1, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1(z_1) w_2(z_2) f(x_1 - h z_1, x_2 - h z_2) dz_1 dz_2. \tag{62}$$

Using the multivariate version of Taylor's theorem derived in Wand and Jones (1995) for this particular application, allows us to rewrite

$$\begin{aligned} f(x_1 - h z_1, x_2 - h z_2) &= f(x_1, x_2) - h(z_1 f_1 + z_2 f_2)(x_1, x_2) \\ &\quad + \frac{1}{2} h^2 (z_1^2 f_{11} + z_1 z_2 (f_{12} + f_{21}) + z_2^2 f_{22})(x_1, x_2) + o(h^2). \end{aligned}$$

We now obtain

$$\begin{aligned}
\mathbb{E} f_{nh}^{++}(x_1, x_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1(z_1) w_2(z_2) \left(f(x_1, x_2) - h(z_1 f_1 + z_2 f_2)(x_1, x_2) \right. \\
&\quad \left. + \frac{1}{2} h^2 (z_1^2 f_{11} + z_1 z_2 (f_{12} + f_{21}) + z_2^2 f_{22})(x_1, x_2) + o(h^2) \right) dz_1 dz_2 \\
&= f(x_1, x_2) - h f_1(x_1, x_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_1 w_1(z_1) w_2(z_2) dz_1 dz_2 \\
&\quad - h f_2(x_1, x_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_2 w_1(z_1) w_2(z_2) dz_1 dz_2 \\
&\quad + \frac{1}{2} h^2 f_{11}(x_1, x_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_1^2 w_1(z_1) w_2(z_2) dz_1 dz_2 \\
&\quad + \frac{1}{2} h^2 (f_{12} + f_{21})(x_1, x_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_1 z_2 w_1(z_1) w_2(z_2) dz_1 dz_2 \\
&\quad + \frac{1}{2} h^2 f_{22}(x_1, x_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_2^2 w_1(z_1) w_2(z_2) dz_1 dz_2 + o(h^2) \\
&= f(x_1, x_2) + \frac{1}{2} h^2 \left(\int_{-\infty}^{\infty} z^2 w_1(z) dz f_{11}(x_1, x_2) + \int_{-\infty}^{\infty} z^2 w_2(z) dz f_{22}(x_1, x_2) \right) + o(h^2).
\end{aligned}$$

This proves statement (22) of the theorem for this individual estimator.

It is easily seen that

$$\begin{aligned}
\mathbb{E} f_{nh}^{--}(x_1, x_2) &= \mathbb{E} f_{nh}^{-+}(x_1, x_2) = \mathbb{E} f_{nh}^{+-}(x_1, x_2) = \mathbb{E} f_{nh}^{++}(x_1, x_2) \\
&= f(x_1, x_2) + \frac{1}{2} h^2 \left(\int_{-\infty}^{\infty} z^2 w_1(z) dz f_{11}(x_1, x_2) + \int_{-\infty}^{\infty} z^2 w_2(z) dz f_{22}(x_1, x_2) \right) + o(h^2)
\end{aligned}$$

and thus

$$\mathbb{E} f_{nh}^{(t)}(x_1, x_2) = f(x_1, x_2) + \frac{1}{2} h^2 \left(\int_{-\infty}^{\infty} z^2 w_1(z) dz f_{11}(x_1, x_2) + \int_{-\infty}^{\infty} z^2 w_2(z) dz f_{22}(x_1, x_2) \right) + o(h^2),$$

proving equation (22).

Next let us derive the asymptotic variance. First, define

$$U_{kh}^{++}(x_1, x_2) := \frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w_1' \left(\frac{x_1 + i - X_{k1}}{h} \right) w_2' \left(\frac{x_2 + j - X_{k2}}{h} \right). \quad (63)$$

Then $f_{nh}^{++}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n U_{kh}^{++}(x_1, x_2)$, and since the terms U_{kh}^{++} are independent,

$$\text{Var}(f_{nh}^{++}(x_1, x_2)) = \frac{1}{n} \text{Var}(U_{1h}^{++}(x_1, x_2)). \quad (64)$$

Secondly, we will determine the variance of $U_{1h}^{++}(x_1, x_2)$. We have

$$\text{Var}(U_{1h}^{++}(x_1, x_2)) = \mathbb{E} U_{1h}^{++}(x_1, x_2)^2 - (\mathbb{E} U_{1h}^{++}(x_1, x_2))^2. \quad (65)$$

Let us begin with determining $\mathbb{E} U_{1h}^{++}(x_1, x_2)^2$. Note that, if $h < \frac{1}{2}$, we have

$$w_1' \left(\frac{x_1 + i_1 - X_{k1}}{h} \right) w_2' \left(\frac{x_2 + i_2 - X_{k2}}{h} \right) w_1' \left(\frac{x_1 + j_1 - X_{k1}}{h} \right) w_2' \left(\frac{x_2 + j_2 - X_{k2}}{h} \right) = 0 \quad (66)$$

unless $i_1 = i_2$ and $j_1 = j_2$, where $i_1, i_2, j_1, j_2 \in \mathbb{Z}$. This holds because if $i_1 \neq i_2$ or $j_1 \neq j_2$, then at least two pairs of arguments in the product (66) are more than distance two apart, rendering the product equal to zero. Thus in the following equation, as $h \rightarrow 0$, only the square products do not vanish and we can write

$$\begin{aligned} \mathbb{E} U_{1h}^{++}(x_1, x_2)^2 &= \mathbb{E} \left(\frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w_1' \left(\frac{x_1 + i - X_{11}}{h} \right) w_2' \left(\frac{x_2 + j - X_{12}}{h} \right) \right)^2 \\ &= \frac{1}{h^8} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{E} \left(w_1' \left(\frac{x_1 + i - X_{11}}{h} \right) w_2' \left(\frac{x_2 + j - X_{12}}{h} \right) \right)^2. \end{aligned}$$

Now we use the substitutions $v_1 := u_1 - i$ and $v_2 := u_2 - j$ to obtain

$$\begin{aligned} \mathbb{E} U_{1h}^{++}(x_1, x_2)^2 &= \frac{1}{h^8} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(w_1' \left(\frac{x_1 + i - u_1}{h} \right) w_2' \left(\frac{x_2 + j - u_2}{h} \right) \right)^2 g(u_1, u_2) du_1 du_2 \\ &= \frac{1}{h^8} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(w_1' \left(\frac{x_1 - v_1}{h} \right) w_2' \left(\frac{x_2 - v_2}{h} \right) \right)^2 g(v_1 + i, v_2 + j) dv_1 dv_2. \end{aligned}$$

Note that the integrand is nonnegative, thus interchanging sums and integrals is allowed (Fubini), so

$$\begin{aligned} \mathbb{E} U_{1h}^{++}(x_1, x_2)^2 &= \frac{1}{h^8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(w_1' \left(\frac{x_1 - v_1}{h} \right) w_2' \left(\frac{x_2 - v_2}{h} \right) \right)^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(v_1 + i, v_2 + j) dv_1 dv_2 \\ &= \frac{1}{h^8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1' \left(\frac{x_1 - v_1}{h} \right)^2 w_2' \left(\frac{x_2 - v_2}{h} \right)^2 F^{++}(v_1, v_2) dv_1 dv_2. \end{aligned}$$

Now apply the substitutions $z_1 = (x_1 - v_1)/h$ and $z_2 = (x_2 - v_2)/h$ and recall the bounded support of w_1' and w_2' . Furthermore, because $\lim_{h \rightarrow 0} F^{++}(x_1 - h z_1, x_2 - h z_2) = F^{++}(x_1, x_2) \leq 1$, we can again apply the Lebesgue dominated convergence theorem

$$\begin{aligned} \mathbb{E} U_{1h}^{++}(x_1, x_2)^2 &= \frac{1}{h^6} \int_{-1}^1 \int_{-1}^1 w_1'(z_1)^2 w_2'(z_2)^2 F^{++}(x_1 - h z_1, x_2 - h z_2) dz_1 dz_2 \\ &= \frac{1}{h^6} F^{++}(x_1, x_2) \int_{-1}^1 w_1'(z_1)^2 dz_1 \int_{-1}^1 w_2'(z_2)^2 dz_2 + o(h^{-6}). \end{aligned} \quad (67)$$

Now note that $E U_{1h}^{++}(x_1, x_2) = E f_{nh}^{++}(x_1, x_2) = f(x_1, x_2) + O(h^2)$. So

$$\begin{aligned}
\text{Var}(f_{nh}^{++}(x_1, x_2)) &= \frac{1}{n} \text{Var}(U_{1h}^{++}(x_1, x_2)) \\
&= \frac{1}{n} \left[E U_{1h}^{++}(x_1, x_2)^2 - (E U_{1h}^{++}(x_1, x_2))^2 \right] \\
&= \frac{1}{n} \left[\frac{1}{h^6} F^{++}(x_1, x_2) \int_{-1}^1 w_1'(z_1)^2 dz_1 \int_{-1}^1 w_2'(z_2)^2 dz_2 + o(h^{-6}) - f(x_1, x_2)^2 - O(h^2) \right] \\
&= \frac{1}{nh^6} F^{++}(x_1, x_2) \int_{-1}^1 w_1'(z_1)^2 dz_1 \int_{-1}^1 w_2'(z_2)^2 dz_2 + o(n^{-1}h^{-6}).
\end{aligned}$$

We can follow a similar procedure to obtain the variances of the other estimators. To summarize we get

$$\begin{aligned}
\text{Var}(f_{nh}^{--}(x_1, x_2)) &= \frac{1}{nh^6} F^{--}(x_1, x_2) \int_{-1}^1 w_1'(z_1)^2 dz_1 \int_{-1}^1 w_2'(z_2)^2 dz_2 + o(n^{-1}h^{-6}), \\
\text{Var}(f_{nh}^{-+}(x_1, x_2)) &= \frac{1}{nh^6} F^{-+}(x_1, x_2) \int_{-1}^1 w_1'(z_1)^2 dz_1 \int_{-1}^1 w_2'(z_2)^2 dz_2 + o(n^{-1}h^{-6}), \\
\text{Var}(f_{nh}^{+-}(x_1, x_2)) &= \frac{1}{nh^6} F^{+-}(x_1, x_2) \int_{-1}^1 w_1'(z_1)^2 dz_1 \int_{-1}^1 w_2'(z_2)^2 dz_2 + o(n^{-1}h^{-6}), \\
\text{Var}(f_{nh}^{++}(x_1, x_2)) &= \frac{1}{nh^6} F^{++}(x_1, x_2) \int_{-1}^1 w_1'(z_1)^2 dz_1 \int_{-1}^1 w_2'(z_2)^2 dz_2 + o(n^{-1}h^{-6}).
\end{aligned}$$

Now let us determine the variance of combinations of these estimators. We have

$$\begin{aligned}
\text{Var}(f_{nh}^{(t)}(x_1, x_2)) &= \text{Var}(t_1 f_{nh}^{--}(x_1, x_2) + t_2 f_{nh}^{-+}(x_1, x_2) + t_3 f_{nh}^{+-}(x_1, x_2) + t_4 f_{nh}^{++}(x_1, x_2)) \\
&= t_1^2 \text{Var}(f_{nh}^{--}(x_1, x_2)) + t_2^2 \text{Var}(f_{nh}^{-+}(x_1, x_2)) + t_3^2 \text{Var}(f_{nh}^{+-}(x_1, x_2)) + t_4^2 \text{Var}(f_{nh}^{++}(x_1, x_2)) \\
&\quad + 2t_1 t_2 \text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{-+}(x_1, x_2)) + 2t_1 t_3 \text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{+-}(x_1, x_2)) \\
&\quad + 2t_1 t_4 \text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{++}(x_1, x_2)) + 2t_2 t_3 \text{Cov}(f_{nh}^{-+}(x_1, x_2), f_{nh}^{+-}(x_1, x_2)) \\
&\quad + 2t_2 t_4 \text{Cov}(f_{nh}^{-+}(x_1, x_2), f_{nh}^{++}(x_1, x_2)) + 2t_3 t_4 \text{Cov}(f_{nh}^{+-}(x_1, x_2), f_{nh}^{++}(x_1, x_2)).
\end{aligned}$$

Let us look at $\text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{-+}(x_1, x_2))$. In similar fashion as we determined the variance, we find

$$\begin{aligned}
\text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{-+}(x_1, x_2)) &= \frac{1}{n} \text{Cov}(U_{1h}^{--}(x_1, x_2), U_{1h}^{-+}(x_1, x_2)) \\
&= \frac{1}{n} [E U_{1h}^{--}(x_1, x_2) U_{1h}^{-+}(x_1, x_2) - E U_{1h}^{--}(x_1, x_2) E U_{1h}^{-+}(x_1, x_2)]
\end{aligned}$$

Let us first determine $E U_{1h}^{--}(x_1, x_2) U_{1h}^{-+}(x_1, x_2)$. Note that, if $h < \frac{1}{2}$, we have

$$w_1' \left(\frac{x_1 - i_1 - X_{k1}}{h} \right) w_2' \left(\frac{x_2 - i_2 - X_{k2}}{h} \right) w_1' \left(\frac{x_1 - j_1 - X_{k1}}{h} \right) w_2' \left(\frac{x_2 + j_2 - X_{k2}}{h} \right) = 0, \quad (68)$$

for all i_1, i_2, j_1 and j_2 . This holds because the second and fourth argument in the product (68) are always more than distance two apart, rendering the product equal to zero. Thus

$$\mathbb{E} U_{1h}^{--}(x_1, x_2) U_{1h}^{++}(x_1, x_2) = 0. \quad (69)$$

Secondly, because we have already determined $\mathbb{E} U_{1h}^{--}(x_1, x_2)$ and $\mathbb{E} U_{1h}^{++}(x_1, x_2)$ earlier, we know that

$$\mathbb{E} U_{1h}^{--}(x_1, x_2) \mathbb{E} U_{1h}^{++}(x_1, x_2) = f(x_1, x_2)^2 + O(h^2). \quad (70)$$

Thus

$$\text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{++}(x_1, x_2)) = \frac{1}{n} [-f(x_1, x_2)^2 - O(h^2)] = o(n^{-1}h^{-6}). \quad (71)$$

This result holds for all the covariances. So we arrive at

$$\begin{aligned} \text{Var}(f_{nh}(x_1, x_2)) &= (t_1^2 F^{--}(x_1, x_2) + t_2^2 F^{++}(x_1, x_2) + t_3^2 F^{+-}(x_1, x_2) + t_4^2 F^{++}(x_1, x_2)) \\ &\quad \frac{1}{nh^6} \int_{-1}^1 w_1'(z_1)^2 dz_1 \int_{-1}^1 w_2'(z_2)^2 dz_2 + o(n^{-1}h^{-6}) \\ &= B(x_1, x_2, t_1, t_2, t_3, t_4) \frac{1}{nh^6} \int_{-1}^1 w_1'(z_1)^2 dz_1 \int_{-1}^1 w_2'(z_2)^2 dz_2 + o(n^{-1}h^{-6}). \end{aligned}$$

This proves statement (23) of the theorem. \square

6.3 Proof of Theorem 4.1

The convex combination of the four density estimators is given by

$$f_{nh}^{(t)}(x_1, x_2) = t_1 f_{nh}^{--}(x_1, x_2) + t_2 f_{nh}^{++}(x_1, x_2) + t_3 f_{nh}^{+-}(x_1, x_2) + t_4 f_{nh}^{++}(x_1, x_2), \quad (72)$$

where $t_1 + t_2 + t_3 + t_4 = 1$. Now define

$$\begin{aligned} S_{1nh}(x_1, x_2) &= f_{nh}^{--}(x_1, x_2) - f_{nh}^{+-}(x_1, x_2), \\ S_{2nh}(x_1, x_2) &= -f_{nh}^{++}(x_1, x_2) + f_{nh}^{+-}(x_1, x_2), \\ S_{3nh}(x_1, x_2) &= f_{nh}^{--}(x_1, x_2) - f_{nh}^{++}(x_1, x_2), \\ S_{4nh}(x_1, x_2) &= -f_{nh}^{+-}(x_1, x_2) + f_{nh}^{++}(x_1, x_2). \end{aligned}$$

We can rewrite (72) as

$$f_{nh}^{(t)}(x_1, x_2) = f_{nh}^{--}(x_1, x_2) - (t_3 + t_4) S_{1nh}(x_1, x_2) - t_2 S_{3nh}(x_1, x_2) + t_4 S_{4nh}(x_1, x_2), \quad (73)$$

Lemma 6.1 *Under the conditions of Theorem 4.1 we have, for $i = 1, \dots, 4$,*

$$\mathbb{E} S_{inh}(x_1, x_2) = 0, \quad (74)$$

$$\mathbb{E} S_{inh}(x_1, x_2)^2 = O\left(\frac{1}{nh^6}\right), \quad (75)$$

$$\mathbb{E} S_{inh}(x_1, x_2)^4 = O\left(\frac{1}{n^2 h^{12}}\right). \quad (76)$$

Proof

We give the proof for $S_{1nh}(x_1, x_2)$. The other claims can be proved similarly. Note that

$$\begin{aligned}
S_{1nh}(x_1, x_2) &= f_{nh}^{--}(x_1, x_2) - f_{nh}^{+-}(x_1, x_2) \\
&= \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_1' \left(\frac{x_1 - i - X_{k1}}{h} \right) w_2' \left(\frac{x_2 - j - X_{k2}}{h} \right) \\
&\quad + \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} w_1' \left(\frac{x_1 + i - X_{k1}}{h} \right) w_2' \left(\frac{x_2 - j - X_{k2}}{h} \right) \\
&= \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} w_1' \left(\frac{x_1 - i - X_{k1}}{h} \right) w_2' \left(\frac{x_2 - j - X_{k2}}{h} \right).
\end{aligned}$$

Define

$$U_{1kh}(x_1, x_2) := \frac{1}{h^4} \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} w_1' \left(\frac{x_1 + i - X_{k1}}{h} \right) w_2' \left(\frac{x_2 + j - X_{k2}}{h} \right). \quad (77)$$

Then $S_{1nh}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n U_{1kh}(x_1, x_2)$ and the terms in the sum are independent.

Following similar steps as in the proof of Theorem 3.1 we get

$$\begin{aligned}
\mathbb{E} U_{1kh}(x_1, x_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 \left(\frac{x_1 - v_1}{h} \right) w_2 \left(\frac{x_2 - v_2}{h} \right) \frac{\partial^2}{\partial v_1 \partial v_2} \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} g(v_1 + i, v_2 + j) dv_1 dv_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 \left(\frac{x_1 - v_1}{h} \right) w_2 \left(\frac{x_2 - v_2}{h} \right) \frac{\partial^2}{\partial v_1 \partial v_2} (1 - F_{Y_2}(v_2)) dv_1 dv_2 = 0.
\end{aligned}$$

We also have, as in the same proof,

$$\mathbb{E} S_{1nh}(x_1, x_2)^2 = \text{Var}(S_{1nh}(x_1, x_2)) = \frac{1}{n} \text{Var}(U_{11h}(x_1, x_2)) = O\left(\frac{1}{nh^6}\right).$$

Finally we consider the fourth moment of $S_{1nh}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n U_{1kh}(x_1, x_2)$. By independence of the terms we have

$$\begin{aligned}
\mathbb{E} S_{1nh}(x_1, x_2)^4 &= \frac{1}{n^3} \mathbb{E} U_{11h}(x_1, x_2)^4 + \frac{3(n-1)}{n^3} \left(\mathbb{E} U_{11h}(x_1, x_2)^2 \right)^2 \\
&= \frac{1}{n^3} O\left(\frac{1}{h^{14}}\right) + \frac{3(n-1)}{n^3} \left(O\left(\frac{1}{h^6}\right) \right)^2 = O\left(\frac{1}{n^2 h^{12}}\right).
\end{aligned}$$

This completes the proof of the lemma. \square

From (73) we get, omitting the arguments (x_1, x_2) ,

$$f_{nh}^{(\hat{t}_n)} - f_{nh}^{(\bar{t})} = -(\hat{t}_{n3} - \bar{t}_3)S_{1nh} - (\hat{t}_{n4} - \bar{t}_4)S_{1nh} - (\hat{t}_{n2} - \bar{t}_2)S_{3nh} + (\hat{t}_{n4} - \bar{t}_4)S_{4nh}. \quad (78)$$

Hence, under the assumptions of the theorem and by the Cauchy Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E} |f_{nh}^{(\hat{t}_n)} - f_{nh}^{(\bar{t})}| &\leq \mathbb{E} |\hat{t}_{n3} - \bar{t}_3| |S_{1nh}| + \mathbb{E} |\hat{t}_{n4} - \bar{t}_4| |S_{1nh}| + \mathbb{E} |\hat{t}_{n2} - \bar{t}_2| |S_{3nh}| + \mathbb{E} |\hat{t}_{n4} - \bar{t}_4| |S_{4nh}| \\
&\leq \left(\mathbb{E} (\hat{t}_{n3} - \bar{t}_3)^2 \right)^{1/2} \left(\mathbb{E} S_{1nh}^2 \right)^{1/2} + \left(\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^2 \right)^{1/2} \left(\mathbb{E} S_{1nh}^2 \right)^{1/2} \\
&\quad + \left(\mathbb{E} (\hat{t}_{n2} - \bar{t}_2)^2 \right)^{1/2} \left(\mathbb{E} S_{3nh}^2 \right)^{1/2} + \left(\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^2 \right)^{1/2} \left(\mathbb{E} S_{4nh}^2 \right)^{1/2} \\
&= \left(o(nh^{10}) O\left(\frac{1}{nh^6}\right) \right)^{1/2} = o(h^2).
\end{aligned}$$

Similarly we have, since $(y_1 + y_2 + y_3 + y_4)^2 \leq 4(y_1^2 + y_2^2 + y_3^2 + y_4^2)$,

$$\begin{aligned}
\text{Var}(f_{nh}^{(\hat{t}_n)} - f_{nh}^{(\bar{t})}) &\leq \mathbb{E} (f_{nh}^{(\hat{t}_n)} - f_{nh}^{(\bar{t})})^2 \\
&\leq 4\mathbb{E} (\hat{t}_{n3} - \bar{t}_3)^2 S_{1nh}^2 + 4\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^2 S_{1nh}^2 + 4\mathbb{E} (\hat{t}_{n2} - \bar{t}_2)^2 S_{3nh}^2 + 4\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^2 S_{4nh}^2 \\
&\leq 4 \left(\mathbb{E} (\hat{t}_{n3} - \bar{t}_3)^4 \right)^{1/2} \left(\mathbb{E} S_{1nh}^4 \right)^{1/2} + 4 \left(\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^4 \right)^{1/2} \left(\mathbb{E} S_{1nh}^4 \right)^{1/2} \\
&\quad + 4 \left(\mathbb{E} (\hat{t}_{n2} - \bar{t}_2)^4 \right)^{1/2} \left(\mathbb{E} S_{3nh}^4 \right)^{1/2} + 4 \left(\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^4 \right)^{1/2} \left(\mathbb{E} S_{4nh}^4 \right)^{1/2} \\
&= o(1) \left(O\left(\frac{1}{n^2 h^{12}}\right) \right)^{1/2} = o\left(\frac{1}{nh^6}\right).
\end{aligned}$$

Since the two bounds above are negligible compared to the order of the bias and variance in Theorem 3.1 it follows that this theorem also holds for the estimator with estimated weights.

In order to prove asymptotic normality note that by Lemma 6.1 and condition (34) it follows that $\sqrt{nh^3}$ times each of the terms in the representation (78) vanish in probability. Also it follows that $\sqrt{nh^3}$ times the expectation of (78) vanishes asymptotically. Hence the limit distributions of $\sqrt{nh^3}(f_{nh}^{(\hat{t}_n)} - \mathbb{E} f_{nh}^{(\hat{t}_n)})$ and $\sqrt{nh^3}(f_{nh}^{(\bar{t})} - \mathbb{E} f_{nh}^{(\bar{t})})$ coincide. The limit distribution of the latter follows by checking the Lyapounov condition for asymptotic normality. \square

6.4 Proof of lemma 4.3

Proof Let us first introduce some notation. Define the vectors $\mathbf{v}(x_1, x_2)$ and $\tilde{\mathbf{v}}_{nh}(x_1, x_2)$ by

$$\begin{aligned}
\mathbf{v}(x_1, x_2) &= (F^{--}(x_1, x_2), F^{-+}(x_1, x_2), F^{+-}(x_1, x_2), F^{++}(x_1, x_2)), \\
\tilde{\mathbf{v}}_{nh}(x_1, x_2) &= (\tilde{F}_{nh}^{--}(x_1, x_2), \tilde{F}_{nh}^{-+}(x_1, x_2), \tilde{F}_{nh}^{+-}(x_1, x_2), \tilde{F}_{nh}^{++}(x_1, x_2)).
\end{aligned}$$

Note that, for n large enough, the components of these vectors are all at least ϵ_n and that they are at most one.

We will only check (29) and (31) for i equal to one. The other cases can be treated similarly. Then we also need the vector of partial derivatives of the the function $\tilde{t}_1(y_1, y_2, y_3, y_4)$. Note that on the line segment between $\tilde{\mathbf{v}}_{nh}(x_1, x_2)$ and $\mathbf{v}(x_1, x_2)$ all the components are all at least ϵ_n and that they are at most one. This implies after some computation

$$\|\nabla \tilde{t}_1(y_1, y_2, y_3, y_4)\|^2 \leq \frac{B}{\epsilon_n^6},$$

for some constant B , for all points (y_1, y_2, y_3, y_4) on this line segment.

We can now apply the multivariate mean value theorem and the Cauchy Schwarz inequality to get

$$\begin{aligned} (\hat{t}_{n1}(x_1, x_2) - \bar{t}_1(x_1, x_2))^2 &= (\tilde{t}_1(\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2)) - \tilde{t}_1(\mathbf{v}(x_1, x_2)))^2 \\ &= (\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2) - \mathbf{v}(x_1, x_2)) \cdot \nabla \tilde{t}_1(y_1, y_2, y_3, y_4)^2 \\ &\leq \|\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2) - \mathbf{v}(x_1, x_2)\|^2 \|\nabla \tilde{t}_1(y_1, y_2, y_3, y_4)\|^2 \\ &\leq \frac{B}{\epsilon_n^6} \|\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2) - \mathbf{v}(x_1, x_2)\|^2, \end{aligned}$$

where (y_1, y_2, y_3, y_4) is a point on the line segment between $\tilde{\mathbf{v}}(x_1, x_2)_{n\tilde{h}}$ and $\mathbf{v}(x_1, x_2)$. Note that $\|\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2) - \mathbf{v}(x_1, x_2)\|^2$ is a sum of four terms like $(\tilde{F}_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^2$, which is smaller than $(F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^2$, and that $E(F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^2$ equals the variance plus the squared bias of $F_{n\tilde{h}}^{--}(x_1, x_2)$. By Theorem 4.2 we can bound these to get

$$E(\hat{t}_{n1}(x_1, x_2) - \bar{t}_1(x_1, x_2))^2 \leq \frac{B}{\epsilon_n^6} \left(O\left(\frac{1}{n\tilde{h}^2}\right) + O(\tilde{h}^4) \right) = O(n^{-2/3}(\log n)^6),$$

for a bandwidth h of order $n^{-1/6}$. This implies that (29) is satisfied.

Let us now check that (31) is satisfied. By an argument similar to the one above it suffices to check if terms like $E(F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^4$ vanish asymptotically. Write

$$F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2) = F_{n\tilde{h}}^{--}(x_1, x_2) - E F_{n\tilde{h}}^{--}(x_1, x_2) + E F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2).$$

By the triangle inequality we have

$$\begin{aligned} \left(E(F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^4 \right)^{1/4} &\leq \\ &\leq \left(E(F_{n\tilde{h}}^{--}(x_1, x_2) - E F_{n\tilde{h}}^{--}(x_1, x_2))^4 \right)^{1/4} + \left(E F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2) \right). \end{aligned}$$

So, by $(a + b)^4 \leq 8(a^4 + b^4)$, $a, b \geq 0$, we also have

$$\begin{aligned} E(F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^4 &\leq \\ &\leq 8E(F_{n\tilde{h}}^{--}(x_1, x_2) - E F_{n\tilde{h}}^{--}(x_1, x_2))^4 + 8\left(E F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2)\right)^4. \end{aligned}$$

Since the bias vanishes by Theorem 4.2 it suffices to prove the bound of the lemma for the fourth power of the error.

Recall from the proof of Theorem 4.2 that $F_{n\tilde{h}}^{--}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n U_{k\tilde{h}}^{--}(x_1, x_2)$, where

$$U_{k\tilde{h}}^{--}(x_1, x_2) := \frac{1}{\tilde{h}^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_1\left(\frac{x_1 - i - X_{k1}}{\tilde{h}}\right) w_2\left(\frac{x_2 - j - X_{k2}}{\tilde{h}}\right).$$

Note that the $U_{k\tilde{h}}^{--}$ are independent. Now write

$$F_{n\tilde{h}}^{--}(x_1, x_2) - E F_{n\tilde{h}}^{--}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n \tilde{U}_{k\tilde{h}}^{--}(x_1, x_2),$$

where $\tilde{U}_{k\tilde{h}}^{--}(x_1, x_2) = U_{k\tilde{h}}^{--}(x_1, x_2) - \mathbb{E} U_{k\tilde{h}}^{--}(x_1, x_2)$. Since $\mathbb{E} \tilde{U}_{k\tilde{h}}^{--}(x_1, x_2)$ equals zero we have

$$\mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n \tilde{U}_{k\tilde{h}}^{--}(x_1, x_2) \right)^4 = \frac{1}{n^3} \mathbb{E} \left(\tilde{U}_{k\tilde{h}}^{--}(x_1, x_2)^4 \right) + 3 \frac{n-1}{n^3} \left(\mathbb{E} \left(\tilde{U}_{k\tilde{h}}^{--}(x_1, x_2)^2 \right) \right)^2.$$

Similar to the derivation of (67) we get

$$\frac{1}{n^3} \mathbb{E} \left(\tilde{U}_{k\tilde{h}}^{--}(x_1, x_2)^4 \right) \sim \frac{1}{n^3} \mathbb{E} \left(U_{k\tilde{h}}^{--}(x_1, x_2)^2 \right) \sim O\left(\frac{1}{n^3 \tilde{h}^4}\right)$$

and

$$3 \frac{n-1}{n^3} \left(\mathbb{E} \left(\tilde{U}_{k\tilde{h}}^{--}(x_1, x_2)^2 \right) \right)^2 = 3 \frac{n-1}{n^3} \left(\text{Var}(U_{k\tilde{h}}^{--}(x_1, x_2)) \right)^2 \sim O\left(\frac{1}{n^2 \tilde{h}^4}\right).$$

Under the condition on \tilde{h} in the lemma both terms vanish. This shows that (31) is satisfied as well. Condition (34) follows from condition (31) by the Cauchy-Schwarz inequality. \square

6.5 An inequality

The next lemma can be used to derive the weights that minimize the asymptotic variance of the convex combination of the original for estimators of the density f .

Lemma 6.2 *Let a_1, \dots, a_m be m positive numbers. Then for all positive t_1, \dots, t_m with $t_1 + \dots + t_m = 1$ we have*

$$a_1 t_1^2 + \dots + a_m t_m^2 \geq \frac{a_1 a_2 \dots a_m}{s_m(a_1, \dots, a_m)}, \quad (79)$$

where $s_m(a_1, \dots, a_m)$ is defined by

$$s_m(a_1, \dots, a_m) = a_2 a_3 \dots a_m + \sum_{j=2}^{m-1} a_1 \dots a_{j-1} a_{j+1} \dots a_m + a_1 a_2 \dots a_{m-1}, \quad (80)$$

the sum of the m products of length $m-1$ obtained by skipping one term in the full product.

The minimum is attained at the t vector given by $t_1 = a_2 a_3 \dots a_m / s_m(a_1, \dots, a_m)$ and $t_m = a_1 a_2 \dots a_{m-1} / s_m(a_1, \dots, a_m)$ and

$$t_i = \frac{a_1 a_2 \dots a_{i-1} a_{i+1} \dots a_m}{s_m(a_1, \dots, a_m)}, \quad i = 2, \dots, m-1.$$

Proof Introduce the inner product $\langle \cdot, \cdot \rangle_a$ and corresponding norm $\|\cdot\|_a$ by

$$\langle x, y \rangle_a = a_2 a_3 \dots a_m x_1 y_1 + a_1 a_3 \dots a_m x_2 y_2 + \dots + a_1 a_2 \dots a_{m-1} x_m y_m, \quad (81)$$

$$\|x\|_a = \left(a_2 a_3 \dots a_m x_1^2 + a_1 a_3 \dots a_m x_2^2 + \dots + a_1 a_2 \dots a_{m-1} x_m^2 \right)^{1/2}. \quad (82)$$

Then, with $\mathbf{1}$ equal to the vector of m ones, the Cauchy-Schwarz inequality implies

$$\begin{aligned}
a_1 a_2 \dots a_m &= (a_1 a_2 \dots a_m)(t_1 + t_2 + \dots + t_m) \\
&= \langle \mathbf{1}, (a_1 t_1, a_2 t_2, \dots, a_m t_m) \rangle_a \leq \| \mathbf{1} \|_a \| (a_1 t_1, a_2 t_2, \dots, a_m t_m) \|_a \\
&= \sqrt{s(a_1, \dots, a_m)} \left(a_2 a_3 \dots a_m (a_1 t_1)^2 + a_1 a_3 \dots a_m (a_2 t_2)^2 + \dots + a_1 a_2 \dots a_{m-1} (a_m t_m)^2 \right)^{1/2} \\
&= \sqrt{s(a_1, \dots, a_m)} \left((a_1 a_2 \dots a_m) (a_1 t_1^2 + a_2 t_2^2 + \dots + a_m t_m^2) \right)^{1/2},
\end{aligned}$$

which implies the inequality after some rewriting. \square

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Bivariate Uniform Deconvolution

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June 9, 2011

Abstract

We construct a density estimator in the bivariate uniform deconvolution model. For this model we derive four inversion formulas to express the bivariate density that we want to estimate in terms of the bivariate density of the observations. By substituting a kernel density estimator of the density of the observations we then get four different estimators. Next we construct an asymptotically optimal convex combination of these four estimators. Expansions for the bias, variance, as well as asymptotic normality, are derived. Some simulated examples are presented.

AMS classification: primary 62G05; secondary 62E20, 62G07, 62G20

Keywords: uniform deconvolution, kernel estimation, bivariate density estimation.

1 Introduction

Before focusing on bivariate deconvolution let us first consider univariate deconvolution. Let X_1, \dots, X_n be i.i.d. observations, where $X_i = Y_i + Z_i$ and Y_i and Z_i are independent. Assume that the unobservable Y_i have distribution function F and density f . Also assume that the unobservable random variables Z_i have a known density k . If the Z_i are uniformly distributed then we have a *uniform deconvolution problem*. Note that the density g of X_i is equal to the convolution of f and k , so $g = k * f$ where $*$ denotes convolution. So we have

$$g(x) = \int_{-\infty}^{\infty} k(x-u)f(u)du. \quad (1)$$

The deconvolution problem is the problem of estimating f or F from the observations X_i .

Several generally applicable methods have been proposed for this deconvolution model. The standard *Fourier type kernel density estimator* for deconvolution problems is based on the Fourier transform, see for instance Wand and Jones (1995). Let w denote a *kernel function* and $h > 0$ a *bandwidth*. The estimator $f_{nh}(x)$ of the density f at the point x is defined as

$$f_{nh}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{\phi_w(ht) \phi_{emp}(t)}{\phi_k(t)} dt = \frac{1}{nh} \sum_{j=1}^n v_h\left(\frac{x - X_j}{h}\right), \quad (2)$$

with

$$v_h(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi_w(s)}{\phi_k(s/h)} e^{-isu} ds, \quad \text{and} \quad \phi_{emp}(t) = \frac{1}{n} \sum_{j=1}^n e^{itX_j},$$

the empirical characteristic function, and ϕ_w and ϕ_k denote the characteristic functions of w and k respectively. An important condition for these estimators to be properly defined is that the characteristic function ϕ_k of the density k has no zeroes, which renders it useless for uniform deconvolution. In fact, Hu and Ridder (2004) argue that in economic applications this assumption is not reasonable. If the error distribution is bounded and symmetric then its characteristic function will have zeros. They propose an approximation of the Fourier transform estimator in such cases. For other modifications of the Fourier inversion method in this problem see Hall and Meister (2007) and Feuerverger, Kim and Sun (2008).

In some univariate deconvolution problems one can apply *nonparametric maximum likelihood*. In the uniform deconvolution problem for instance the error Z is Uniform[0,1) distributed. So in this particular deconvolution problem we assume to have i.i.d. observations from the density

$$g(x) = \int_{-\infty}^{\infty} I_{[0,1)}(x - u) f(u) du = \int_{x-1}^x f(u) du = F(x) - F(x - 1). \quad (3)$$

Groeneboom and Jongbloed (2003) consider density estimation in this problem. They propose a kernel density estimator based on the nonparametric maximum likelihood estimator (NPMLE) of the distribution function F and derive its asymptotic properties. For estimators of the distribution function in uniform deconvolution, related to the NPMLE, we refer to Groeneboom and Wellner (1992) and Van Es and Van Zuijlen (1996).

A selected group of deconvolution problems allows explicit *inversion formulas* of (1) expressing the density of interest f in terms of the density g of the data. In these cases we can estimate f by substituting for instance a direct kernel density estimate of g in the inversion formula. In Van Es and Kok (1998) this strategy has been pursued for deconvolution problems where k equals the exponential density, the Laplace density, and their repeated convolutions.

If we apply inversion to the uniform problem then it turns out we get two obvious inversion formulas. Of course these inversions agree on the set of densities of the form (3), but they are different outside of this set. Plugging in a kernel estimator of the density g of the observations, which is typically *not* of this form, then yields two estimators of f . These can then in some sense be optimally combined in a convex combination. This approach is developed in Van Es (2010). Here we will follow this approach in the bivariate uniform deconvolution setting.

Let us now consider *bivariate deconvolution*. The bivariate convolution formula $\mathbf{X}_i = \mathbf{Y}_i + \mathbf{Z}_i$, where \mathbf{X}_i , \mathbf{Y}_i and \mathbf{Z}_i stand for two dimensional random vectors, can be written in vector notation as

$$\begin{pmatrix} X_{i1} \\ X_{i2} \end{pmatrix} = \begin{pmatrix} Y_{i1} \\ Y_{i2} \end{pmatrix} + \begin{pmatrix} Z_{i1} \\ Z_{i2} \end{pmatrix}. \quad (4)$$

The estimation principles described above can in principle all be attempted in the bivariate problem as well. See for instance Youndjé and Wells (2008) for recent results on multivariate Fourier type kernel deconvolution. Approaches based on nonparametric maximum likelihood and inversion hardly exist to our knowledge.

In the bivariate uniform deconvolution setting the random vector \mathbf{Z}_i has a $\text{Uniform}([0, 1] \times [0, 1])$ distribution, i.e. it is uniformly distributed on the unit square. Here we can also express the bivariate density g of the observations in terms of the bivariate distribution function F , with density f , of the random vector \mathbf{Y} . We have

$$\begin{aligned} g(x_1, x_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_{[0,1]}(x_1 - u_1) I_{[0,1]}(x_2 - u_2) f(u_1, u_2) du_1 du_2 \\ &= \int_{x_2-1}^{x_2} \int_{x_1-1}^{x_1} f(u_1, u_2) du_1 du_2 \\ &= F(x_1, x_2) - F(x_1, x_2 - 1) - F(x_1 - 1, x_2) + F(x_1 - 1, x_2 - 1). \end{aligned} \quad (5)$$

This is the bivariate analogue of formula (3). Note that, again, the Fourier inversion approach can not be used because of the zeros in the characteristic function of the bivariate uniform distribution.

The main aim of this paper is to develop the inversion approach of Van Es (2010) for bivariate uniform deconvolution. In Chapter 2 we derive four inversion formulas for (5). This yields the same number of possible estimators if we plug in a density estimator of the density g of the observations. In Chapter 3 we combine these estimators in a convex combination which is asymptotically optimal in some sense. The weights of this combination turn out to depend on the unknown distribution F . A general theorem for an estimator with estimated weights is given in Chapter 4. We also present specific estimators of these weights. Simulated examples are presented in Chapter 5. Chapter 6 contains the proofs.

2 Inversion formulas

Recall that the density of the \mathbf{Z}_i is equal to $k(z_1, z_2) = I_{[0,1] \times [0,1]}(z_1, z_2) = I_{[0,1]}(z_1) I_{[0,1]}(z_2)$. This yields formula (5) which expresses $g(x_1, x_2)$ in terms of $F(x_1, x_2)$. Lemma 2.1 below demonstrates that the converse is also feasible.

First note that for

$$\begin{aligned} F^{--}(y_1, y_2) &:= \Pr(Y_1 \leq y_1, Y_2 \leq y_2), \\ F^{-+}(y_1, y_2) &:= \Pr(Y_1 \leq y_1, Y_2 > y_2), \\ F^{+-}(y_1, y_2) &:= \Pr(Y_1 > y_1, Y_2 \leq y_2), \\ F^{++}(y_1, y_2) &:= \Pr(Y_1 > y_1, Y_2 > y_2). \end{aligned}$$

the following equalities hold

$$F^{--}(x_1, x_2) = F(x_1, x_2), \quad (6)$$

$$F^{-+}(x_1, x_2) = F_{Y_1}(x_1) - F(x_1, x_2), \quad (7)$$

$$F^{+-}(x_1, x_2) = F_{Y_2}(x_2) - F(x_1, x_2), \quad (8)$$

$$F^{++}(x_1, x_2) = F(x_1, x_2) - F_{Y_1}(x_1) - F_{Y_2}(x_2) + 1. \quad (9)$$

If we know $F(x_1, x_2)$ and if this function is continuously differentiable over x_1 and x_2 , then we know $f(x_1, x_2)$, because $f(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2)$. In fact, combined with the formulas above, and (5), this gives us four different inversion formulas to obtain f and F from g , as is stated in the following Lemma.

Lemma 2.1 *Assume that $\lim_{x_1 \rightarrow \pm\infty} f(x_1, x_2) = 0$ and $\lim_{x_2 \rightarrow \pm\infty} f(x_1, x_2) = 0$. Then we have*

$$F^{--}(x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g(x_1 - i, x_2 - j), \quad (10)$$

$$F^{-+}(x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} g(x_1 - i, x_2 + j), \quad (11)$$

$$F^{+-}(x_1, x_2) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} g(x_1 + i, x_2 - j), \quad (12)$$

$$F^{++}(x_1, x_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(x_1 + i, x_2 + j). \quad (13)$$

Furthermore, assuming that $g(x_1, x_2)$ is twice mixed continuously differentiable over x_1 and x_2 , then there are four inversion formulas to recover f from g . We have

$$f(x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1 - i, x_2 - j), \quad (14)$$

$$f(x_1, x_2) = - \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1 - i, x_2 + j), \quad (15)$$

$$f(x_1, x_2) = - \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1 + i, x_2 - j), \quad (16)$$

$$f(x_1, x_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1 + i, x_2 + j). \quad (17)$$

To get some more insight in these inversion formulas note that (5) can be interpreted as a probability for \mathbf{Y} (under F). We have

$$g(x_1, x_2) = P_F(\mathbf{Y} \in (x_1 - 1, x_1] \times (x_2 - 1, x_2]).$$

So $g(x_1, x_2)$ is equal to the probability that \mathbf{Y} belongs to a specific square $(x_1-1, x_1] \times (x_2-1, x_2]$. Adding up over suitable squares we then get the probability that \mathbf{Y} belongs to a specific quadrant with a given vertex. For a formal proof see Chapter 6.

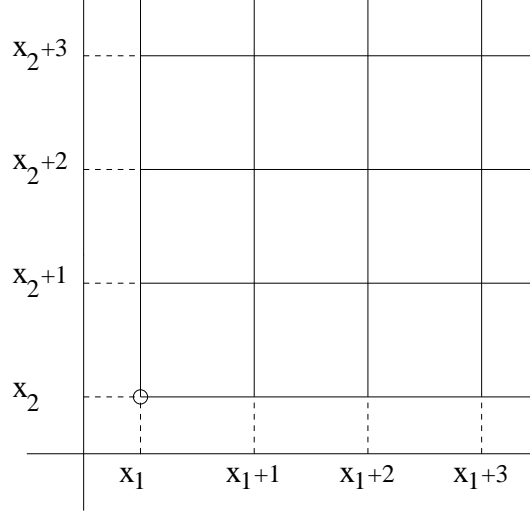


Figure 1: $F^{++}(x_1, x_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_F(\mathbf{Y}_i \in (x_1 + i - 1, x_1 + i] \times (x_2 + j - 1, x_2 + j])$.

3 Estimation of the density function

In the previous chapter we have derived inversion formulas that express the density f in terms of the density g of the observations. Now we can use an estimator of g , for which we have observations, to estimate f . For an arbitrary density that is not of the form (5), the inversions will in general not yield distribution functions or densities, nor will they coincide. This typically happens if we estimate g .

We use kernel smoothing but of course other estimators can be used as well. Let us introduce a bivariate kernel density estimator with bivariate kernel function \mathbf{w} and bandwidth $h > 0$. The estimator g_{nh} of g is given by

$$g_{nh}(x_1, x_2) = \frac{1}{nh^2} \sum_{k=1}^n \mathbf{w}\left(\frac{x_1 - X_{k1}}{h}, \frac{x_2 - X_{k2}}{h}\right). \quad (18)$$

Usually, \mathbf{w} is chosen to be a bivariate probability density function. This way it is ensured that g_{nh} is also a density. See for instance Silverman (1986) and Wand and Jones (1995).

We impose the following condition on the kernel function.

Condition W

The function \mathbf{w} is a probability density function on \mathbb{R}^2 with support $[-1, 1] \times [-1, 1]$. Furthermore, we will use a product kernel $\mathbf{w}(u_1, u_2) = w(u_1)w(u_2)$, where $w(u_i)$, with $i \in \{1, 2\}$, denotes a continuously differentiable univariate symmetric probability density function.

Plugging in the kernel estimator in the four inversion formulas of Lemma 2.1 we get four kernel estimators of the density given by

$$\begin{aligned}
f_{nh}^{--}(x_1, x_2) &= \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w' \left(\frac{x_1 - i - X_{k1}}{h} \right) w' \left(\frac{x_2 - j - X_{k2}}{h} \right), \\
f_{nh}^{-+}(x_1, x_2) &= -\frac{1}{nh^4} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} w' \left(\frac{x_1 - i - X_{k1}}{h} \right) w' \left(\frac{x_2 + j - X_{k2}}{h} \right), \\
f_{nh}^{+-}(x_1, x_2) &= -\frac{1}{nh^4} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} w' \left(\frac{x_1 + i - X_{k1}}{h} \right) w' \left(\frac{x_2 - j - X_{k2}}{h} \right), \\
f_{nh}^{++}(x_1, x_2) &= \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w' \left(\frac{x_1 + i - X_{k1}}{h} \right) w' \left(\frac{x_2 + j - X_{k2}}{h} \right).
\end{aligned}$$

We will derive $f_{nh}^{++}(x_1, x_2)$. The other three estimators follow similarly. Define $w'(u) := \frac{d}{du}w(u)$. Lemma 2.1 in combination with $\frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2) = f(x_1, x_2)$ gives us

$$\begin{aligned}
f_{nh}^{++}(x_1, x_2) &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g_{nh}(x_1 + i, x_2 + j) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{\partial^2}{\partial x_1 \partial x_2} \frac{1}{n} \sum_{k=1}^n \frac{1}{h^2} w \left(\frac{x_1 + i - X_{k1}}{h}, \frac{x_2 + j - X_{k2}}{h} \right) \right) \\
&= \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w' \left(\frac{x_1 + i - X_{k1}}{h} \right) w' \left(\frac{x_2 + j - X_{k2}}{h} \right).
\end{aligned}$$

Note that, because of the bounded support of w , the sum is in fact a finite sum. In the last step we used the fact that w is a product kernel, and thus $\frac{\partial^2}{\partial u_1 \partial u_2} w(u_1, u_2) = w'(u_1)w'(u_2)$.

Next we introduce a convex combination of the four previous estimators. Write

$$f_{nh}^{(t)}(x_1, x_2) = t_1 f_{nh}^{--}(x_1, x_2) + t_2 f_{nh}^{-+}(x_1, x_2) + t_3 f_{nh}^{+-}(x_1, x_2) + t_4 f_{nh}^{++}(x_1, x_2), \quad (19)$$

where $t = (t_1, t_2, t_3, t_4)$ and $t_1 + t_2 + t_3 + t_4 = 1$. For suitable choices of t_1, t_2, t_3, t_4 this combination will turn out to have better properties than any of the estimators separately. Notice that when we set t_1, t_2, t_3 , or t_4 equal to one and the others equal to zero, we get results for $f_{nh}^{--}, f_{nh}^{-+}, f_{nh}^{+-}$, or f_{nh}^{++} individually.

Theorem 3.1 *Assume that Condition W is satisfied, that f is bounded, and that $\lim_{x_1 \rightarrow \pm\infty} f(x_1, x_2) = \lim_{x_2 \rightarrow \pm\infty} f(x_1, x_2) = 0$. If f is twice continuously differentiable on a neighborhood of $x = (x_1, x_2)$ then, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$, we have*

$$E f_{nh}^{(t)}(x_1, x_2) = f(x_1, x_2) + \frac{1}{2} h^2 \int_{-\infty}^{\infty} z_1^2 w(z) dz (f_{11} + f_{22})(x_1, x_2) + o(h^2). \quad (20)$$

Furthermore, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$, we have

$$\text{Var}(f_{nh}^{(t)}(x_1, x_2)) = \frac{1}{nh^6} B(x_1, x_2, t_1, t_2, t_3, t_4) \left(\int_{-1}^1 w'(z)^2 dz \right)^2 + o(n^{-1}h^{-6}) \quad (21)$$

where

$$B(x_1, x_2, t_1, t_2, t_3, t_4) = (t_1^2 F^{--} + t_2^2 F^{-+} + t_3^2 F^{+-} + t_4^2 F^{++})(x_1, x_2). \quad (22)$$

From the theorem we see that the expectation of $f_{nh}^{(t)}(x_1, x_2)$ is the same whatever convex combination we choose for. Lemma 3.2 gives the weights that minimize the leading term in the variance (21).

Lemma 3.2 *Assume that (x_1, x_2) is an interior point of the support of f . The weights t_1, t_2, t_3 and t_4 , with $t_1 + t_2 + t_3 + t_4 = 1$, that minimize the leading term in the variance (21), are denoted by $\bar{t}_1(x_1, x_2), \bar{t}_2(x_1, x_2), \bar{t}_3(x_1, x_2)$ and $\bar{t}_4(x_1, x_2)$ and they are equal to*

$$\begin{aligned} \bar{t}_1(x_1, x_2) &= F^{-+,+-,++}(x_1, x_2)A(x_1, x_2), \\ \bar{t}_2(x_1, x_2) &= F^{--,+-,++}(x_1, x_2)A(x_1, x_2), \\ \bar{t}_3(x_1, x_2) &= F^{--, -+, ++}(x_1, x_2)A(x_1, x_2), \\ \bar{t}_4(x_1, x_2) &= F^{--, -+, +-}(x_1, x_2)A(x_1, x_2). \end{aligned}$$

The resulting variance of this optimal convex combination is then equal to

$$\text{Var}(f_{nh}(x_1, x_2)) = A(x_1, x_2)C(x_1, x_2)\frac{1}{nh^6}\left(\int_{-1}^1 w'(z)^2 dz\right)^2 + o(n^{-1}h^{-6}), \quad (23)$$

Here

$$A(x_1, x_2) := (F^{-+,+-,++} + F^{--,+-,++} + F^{--, -+, ++} + F^{--, -+, +-})^{-1}(x_1, x_2). \quad (24)$$

where, for $a_1, a_2, b_1, b_2, c_1, c_2 \in \{-, +\}$,

$$F^{a_1 a_2, b_1 b_2, c_1 c_2}(x_1, x_2) := F^{a_1 a_2}(x_1, x_2)F^{b_1 b_2}(x_1, x_2)F^{c_1 c_2}(x_1, x_2), \quad (25)$$

and

$$C(x_1, x_2) := F^{--}(x_1, x_2)F^{-+}(x_1, x_2)F^{+-}(x_1, x_2)F^{++}(x_1, x_2). \quad (26)$$

Proof

First note that the weights are well defined since the fact that (x_1, x_2) is an interior point of the support of f implies that $F^{--}(x_1, x_2), F^{-+}(x_1, x_2), F^{+-}(x_1, x_2)$ and $F^{++}(x_1, x_2)$ are strictly positive. The lower bound now follows from Lemma 6.2 in Chapter 6. \square

Note that in general, of course, we do not know F . However, in Section 4 we show that we can estimate $F^{--}(x_1, x_2), F^{-+}(x_1, x_2), F^{+-}(x_1, x_2)$, and $F^{++}(x_1, x_2)$, again using the inversion formulas of Theorem 2.1. This will lead to estimates of the optimal weights. We then prove that the estimator with estimated weights shares the properties of Theorem 3.1 with the optimal weights.

4 The final estimator with estimated optimal weights

Let us write $\hat{t}_n(x_1, x_2) = (\hat{t}_{n1}(x_1, x_2), \dots, \hat{t}_{n4}(x_1, x_2))$ for a vector of estimated weights. The next theorem shows that under some conditions on these estimators the limit behaviour of $f_{nh}^{(\hat{t}_n)}(x_1, x_2)$ resembles the optimal limit behaviour of the estimator $f_{nh}^{(\bar{t})}(x_1, x_2)$.

Theorem 4.1 *Assume that Condition W is satisfied, that f is bounded, and that $\lim_{x_1 \rightarrow \pm\infty} f(x_1, x_2) = \lim_{x_1 \rightarrow \pm\infty} f(x_1, x_2) = 0$. Assume for $i = 1, \dots, 4$,*

$$E(\hat{t}_{ni}(x_1, x_2) - \bar{t}_i(x_1, x_2))^2 = o(nh^{10}). \quad (27)$$

If f is twice continuously differentiable on a neighborhood of $x = (x_1, x_2)$ then, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$, we have

$$E f_{nh}^{(\hat{t}_n)}(x_1, x_2) = f(x_1, x_2) + \frac{1}{2}h^2 \int_{-\infty}^{\infty} z_1^2 w(z) dz (f_{11} + f_{22})(x_1, x_2) + o(h^2). \quad (28)$$

Assume for $i = 1, \dots, 4$,

$$E(\hat{t}_{ni}(x_1, x_2) - \bar{t}_i(x_1, x_2))^4 = o(1). \quad (29)$$

Then, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$, we have

$$\text{Var}(f_{nh}^{(\hat{t}_n)}(x_1, x_2)) = \frac{1}{nh^6} \sigma(x_1, x_2)^2 + o(n^{-1}h^{-6}), \quad (30)$$

where, with the notation of Lemma 3.2, $\sigma(x_1, x_2)^2$ is defined by

$$\sigma(x_1, x_2)^2 = A(x_1, x_2)C(x_1, x_2) \left(\int_{-1}^1 w'(z)^2 dz \right)^2. \quad (31)$$

Assume for $i = 1, \dots, 4$,

$$E(\hat{t}_{ni}(x_1, x_2) - \bar{t}_i(x_1, x_2))^2 = o(1). \quad (32)$$

Then the estimator is asymptotically normally distributed. We have, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$,

$$\sqrt{nh}^3 \left(f_{nh}^{(\hat{t}_n)}(x_1, x_2) - E f_{nh}^{(\hat{t}_n)}(x_1, x_2) \right) \xrightarrow{\mathcal{D}} N(0, \sigma(x_1, x_2)^2). \quad (33)$$

Let us next construct suitable estimators of the weights based on the estimators of F^{--}, F^{+-}, F^{++} and F^{++} . As in estimation of the density we can plug in (18) into the inversion formulas for F in Lemma 2.1 and get kernel estimators of $F^{--}(x_1, x_2), F^{+-}(x_1, x_2), F^{++}(x_1, x_2)$ and $F^{++}(x_1, x_2)$.

We get four estimators, given by

$$\begin{aligned}
F_{nh}^{--}(x_1, x_2) &= \frac{1}{nh^2} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w\left(\frac{x_1 - i - X_{k1}}{h}\right) w\left(\frac{x_2 - j - X_{k2}}{h}\right), \\
F_{nh}^{-+}(x_1, x_2) &= \frac{1}{nh^2} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} w\left(\frac{x_1 - i - X_{k1}}{h}\right) w\left(\frac{x_2 + j - X_{k2}}{h}\right), \\
F_{nh}^{+-}(x_1, x_2) &= \frac{1}{nh^2} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} w\left(\frac{x_1 + i - X_{k1}}{h}\right) w\left(\frac{x_2 - j - X_{k2}}{h}\right), \\
F_{nh}^{++}(x_1, x_2) &= \frac{1}{nh^2} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w\left(\frac{x_1 + i - X_{k1}}{h}\right) w\left(\frac{x_2 + j - X_{k2}}{h}\right). \tag{34}
\end{aligned}$$

The following theorem establishes the asymptotic bias and variance of these four estimators.

Theorem 4.2 *Assume that Condition W is satisfied. Then, as $n \rightarrow \infty, h \rightarrow 0, nh \rightarrow \infty$, we have We have*

$$\begin{aligned}
E F_{nh}^{--}(x_1, x_2) &= F^{--}(x_1, x_2) + \frac{1}{2}h^2(F_{11}^{--} + F_{22}^{--})(x_1, x_2) \int_{-1}^1 z^2 w(z) dz + o(h^2), \\
E F_{nh}^{-+}(x_1, x_2) &= F^{-+}(x_1, x_2) + \frac{1}{2}h^2(F_{11}^{-+} + F_{22}^{-+})(x_1, x_2) \int_{-1}^1 z^2 w(z) dz + o(h^2), \\
E F_{nh}^{+-}(x_1, x_2) &= F^{+-}(x_1, x_2) + \frac{1}{2}h^2(F_{11}^{+-} + F_{22}^{+-})(x_1, x_2) \int_{-1}^1 z^2 w(z) dz + o(h^2), \\
E F_{nh}^{++}(x_1, x_2) &= F^{++}(x_1, x_2) + \frac{1}{2}h^2(F_{11}^{++} + F_{22}^{++})(x_1, x_2) \int_{-1}^1 z^2 w(z) dz + o(h^2).
\end{aligned}$$

where $F_{11}^{--} = \frac{\partial^2 F^{--}(x_1, x_2)}{\partial x_1^2}$ and $F_{22}^{--} = \frac{\partial^2 F^{--}(x_1, x_2)}{\partial x_2^2}$, etc..

For the variances we have

$$\begin{aligned}
\text{Var}(F_{nh}^{--}(x_1, x_2)) &= F^{--}(x_1, x_2) \frac{1}{nh^2} \left(\int_{-1}^1 w^2(z) dz \right)^2 + o\left(\frac{1}{nh^2}\right), \\
\text{Var}(F_{nh}^{-+}(x_1, x_2)) &= F^{-+}(x_1, x_2) \frac{1}{nh^2} \left(\int_{-1}^1 w^2(z) dz \right)^2 + o\left(\frac{1}{nh^2}\right), \\
\text{Var}(F_{nh}^{+-}(x_1, x_2)) &= F^{+-}(x_1, x_2) \frac{1}{nh^2} \left(\int_{-1}^1 w^2(z) dz \right)^2 + o\left(\frac{1}{nh^2}\right), \\
\text{Var}(F_{nh}^{++}(x_1, x_2)) &= F^{++}(x_1, x_2) \frac{1}{nh^2} \left(\int_{-1}^1 w^2(z) dz \right)^2 + o\left(\frac{1}{nh^2}\right).
\end{aligned}$$

For the proof of this theorem see Chapter 6.

Next we write the optimal weights of Lemma 3.2 in terms of functions \tilde{t}_i defined by

$$\bar{t}_i(x_1, x_2) = \tilde{t}_i(F^{--}(x_1, x_2), F^{-+}(x_1, x_2), F^{+-}(x_1, x_2), F^{++}(x_1, x_2)), \quad i = 1, \dots, 4.$$

Let (ϵ_n) denote a sequence of numbers with $0 < \epsilon_n < 1$ and $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Then define truncated versions of the estimators $F_{nh}^{--}(x_1, x_2), F_{nh}^{-+}(x_1, x_2), F_{nh}^{+-}(x_1, x_2), F_{nh}^{++}(x_1, x_2)$ and $F_{nh}^{++}(x_1, x_2)$ by

$$\begin{aligned}\tilde{F}_{nh}^{--}(x_1, x_2) &= \min(\max(F_{nh}^{--}(x_1, x_2), \epsilon_n), 1), \\ \tilde{F}_{nh}^{-+}(x_1, x_2) &= \min(\max(F_{nh}^{-+}(x_1, x_2), \epsilon_n), 1), \\ \tilde{F}_{nh}^{+-}(x_1, x_2) &= \min(\max(F_{nh}^{+-}(x_1, x_2), \epsilon_n), 1), \\ \tilde{F}_{nh}^{++}(x_1, x_2) &= \min(\max(F_{nh}^{++}(x_1, x_2), \epsilon_n), 1).\end{aligned}$$

Since the bandwidth used in the estimators of the weights can in general be different to the bandwidth h used in the estimator of f , we will denote this bandwidth by \tilde{h} . We now obtain estimators of the weights by plugging in these estimators. We get

$$\hat{t}_{ni}(x_1, x_2) = \tilde{t}_i(\tilde{F}_{n\tilde{h}}^{--}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{-+}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{+-}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{++}(x_1, x_2)), \quad i = 1, \dots, 4.$$

The next lemma shows that these estimators, with a suitable bandwidth, can be used to estimate the optimal weights without disturbing the asymptotics of Theorem 3.1.

Lemma 4.3 *If $h \gg n^{-1/6}$, $\epsilon_n = 1/\log n$, and if we use a bandwidth \tilde{h} of the form $\tilde{h} = cn^{-1/6}$, where c is a constant, then the estimators*

$$\hat{t}_{ni}(x_1, x_2) = \tilde{t}_i(\tilde{F}_{n\tilde{h}}^{--}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{-+}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{+-}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{++}(x_1, x_2))$$

satisfy (27), (29) and (32).

If we compare the performance of our final estimator with estimated optimal weights to the performance of the four individual estimators then we see that the first order of the expectation is the same. The variance of the combined estimator contains the term $C(x_1, x_2)$ which is equal to the product of $F^{--}(x_1, x_2), F^{-+}(x_1, x_2), F^{+-}(x_1, x_2)$ and $F^{++}(x_1, x_2)$. This shows that the variance is small along the edge of the support of f . By Theorem 3.1 the variance of, for instance, $f_{nh}^{--}(x_1, x_2)$ is proportional to $F^{--}(x_1, x_2)$. This shows that this estimator will perform better in the lower left of the support of f than it will in the other part. By using the estimated optimal convex combination the worse behavior of the four individual estimators in certain areas is reduced.

If we minimize the pointwise asymptotic mean squared error of $f_{nh}^{(\hat{t}_n)}(x_1, x_2)$ and thus balance its asymptotic squared bias and its asymptotic variance given by Theorem 4.1 then we see that the optimal bandwidth is of order $n^{-1/10}$. The corresponding mean squared error is then equal to $n^{-2/5}$. This of course raises the problem of bandwidth selection which we will not pursue here.

5 Simulated examples

To illustrate the estimator we have simulated two examples. In the first example the density f is unimodal. In the second example f is a mixture of two unimodal bivariate densities, rendering it bimodal. In the first example f is concentrated on the square $[0.25, 1.75] \times [0.25, 1.75]$. In the second example f is concentrated on the square $[0.2, 1.8] \times [0.2, 1.8]$. This means that both deconvolution problems are not at all trivial.

To speed up computations we have followed the bivariate binning technique as advised in Wand (1994). For the x and y coordinates we have chosen for a grid of 500 points between -1 and 4. We have used a product kernel based on the so called biweight kernel given by

$$w(u) = \frac{15}{16} (1 - u^2)^2 I_{[-1,1]}(u). \quad (35)$$

Example 5.1 In our first example f is the density of the random vector (Y_1, Y_2) , where Y_1 and Y_2 are two independent random variables that each have a certain shifted and rescaled beta distribution. To be more specific $Y_i = 0.25 + 1.5V_i, i = 1, 2$, where the V_i are independent and both Beta(3,3) distributed. We have simulated 1000 values so $n = 1000$. The bandwidth h , chosen by hand, is equal to 0.5.

The true density f and its estimate are given in Figure 2. The difference between the true density and the estimate is plotted in Figure 3. The right plot in Figure 3 shows f_{nh}^{+-} . Clearly this estimate is best in the $+-$ quadrant, as predicted by the theory.

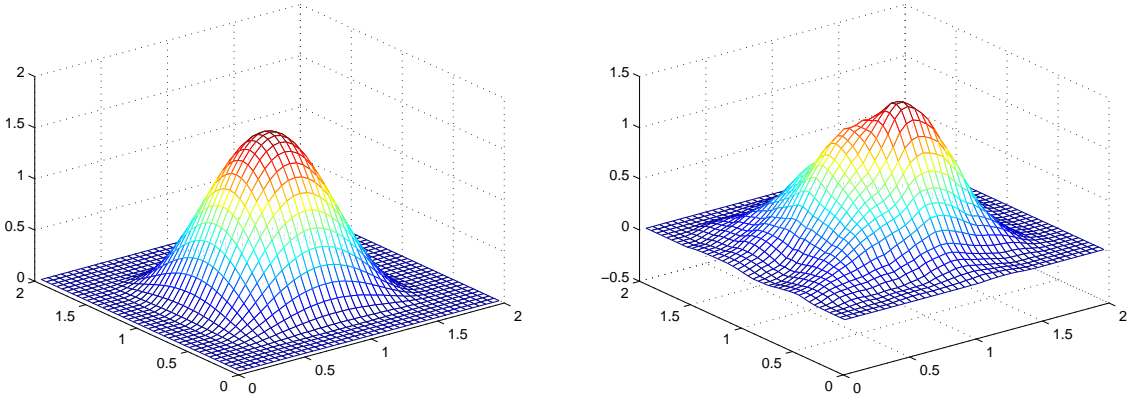


Figure 2: Left: the true density. Right: the estimate.

Example 5.2 In our second example f is the density of the random vector (Y_1, Y_2) , where Y_1 and Y_2 are dependent random variables with a bimodal distribution. The distribution of the vector is a mixture of two distributions like the one in Example 5.1. The values of the Y 's are generated as follows. With V_1 and V_2 having the same distribution as in the previous example

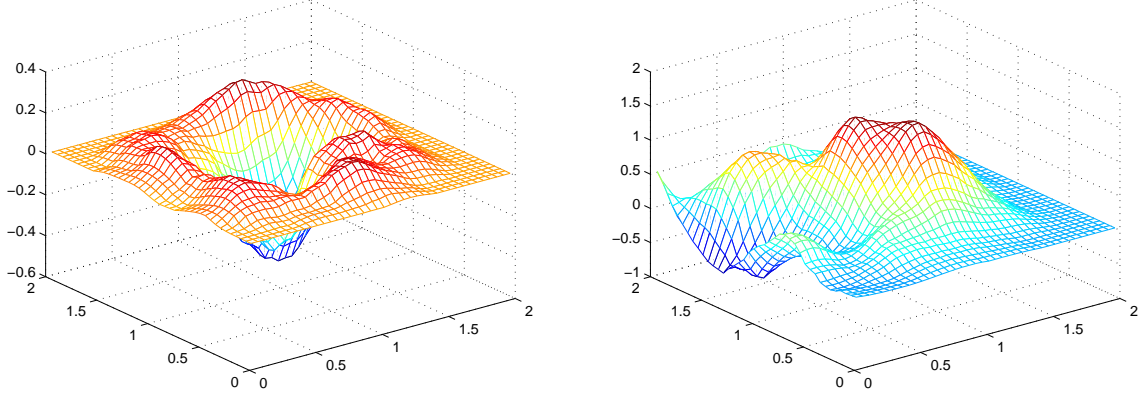


Figure 3: Left: the difference of the true density and the estimate. Right: f_{nh}^{+-} .

the Y values are given by

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} V_1 + 0.2 \\ V_2 + 0.8 \end{pmatrix} & , \text{ with probability } 2/5, \\ \begin{pmatrix} V_1 + 0.8 \\ V_2 + 0.2 \end{pmatrix} & , \text{ with probability } 3/5. \end{cases}$$

We have simulated 5000 values so $n = 5000$. The bandwidth h , chosen by hand, is equal to 0.35.

The true density f and its estimate are given in Figure 4. The difference between the true density and the estimate is plotted in Figure 5. The right plot in Figure 5 shows f_{nh}^{+-} . Clearly this estimate is best in the $-+$ quadrant, as predicted by the theory.

6 Proofs

6.1 Proof of Lemma 2.1

Let us first determine the inversion formulas for $F(x_1, x_2)$. We sum $g(x_1 - i, x_2) = F(x_1 - i, x_2) - F(x_1 - i, x_2 - 1) - F(x_1 - i - 1, x_2) + F(x_1 - i - 1, x_2 - 1)$ over the first coordinate to

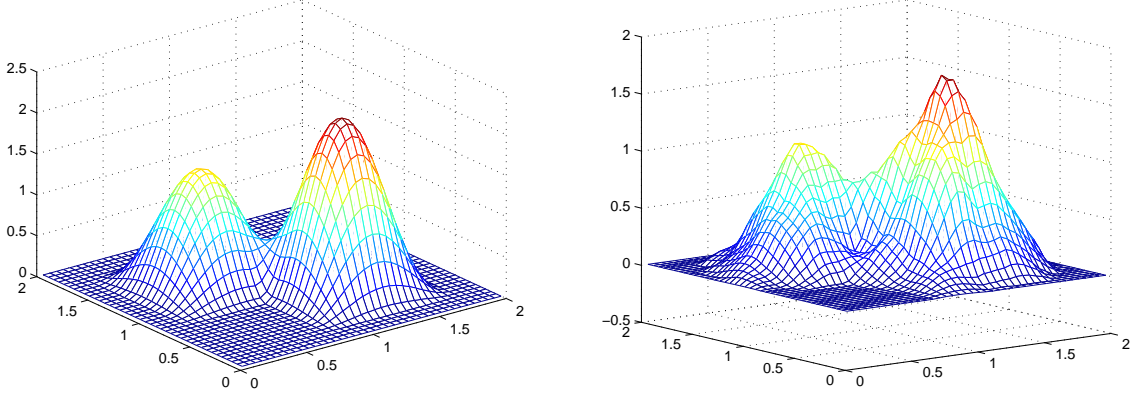


Figure 4: $n = 5000, h = 0.35$. Left: the true density. Right: the estimate.

obtain two telescopic sums. Thus we get

$$\begin{aligned}
& \sum_{i=0}^{\infty} g(x_1 - i, x_2) \\
&= \sum_{i=0}^{\infty} \{F(x_1 - i, x_2) - F(x_1 - i, x_2 - 1) - F(x_1 - i - 1, x_2) + F(x_1 - i - 1, x_2 - 1)\} \\
&= \sum_{i=0}^{\infty} \{F(x_1 - i, x_2) - F(x_1 - i - 1, x_2)\} - \sum_{i=0}^{\infty} \{F(x_1 - i, x_2 - 1) - F(x_1 - i - 1, x_2 - 1)\} \\
&= F(x_1, x_2) - F(x_1, x_2 - 1).
\end{aligned} \tag{36}$$

Here we used that $\lim_{i \rightarrow \infty} F(x_1 - i, x_2) = \lim_{i \rightarrow \infty} F(x_1 - i, x_2 - 1) = 0$, for F is a bivariate distribution function. Next, we sum over the second coordinate. Because we also have $\lim_{j \rightarrow \infty} F(x_1, x_2 - j) = 0$, we get

$$\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} g(x_1 - i, x_2 - j) = \sum_{j=0}^{\infty} \{F(x_1, x_2 - j) - F(x_1, x_2 - j - 1)\} = F(x_1, x_2). \tag{37}$$

Because the terms are nonnegative, the order of summation can be interchanged and we have shown (10). Thus we have found an expression for the unobservable probability distribution function F in terms of the observable density function g .

Above, we iterated over $-i$, so now let us determine what happens when we iterate over $+i$. First, we write $g(x_1 + i, x_2)$ as

$$g(x_1 + i, x_2) = F(x_1 + i, x_2) - F(x_1 + i, x_2 - 1) - F(x_1 + i - 1, x_2) + F(x_1 + i - 1, x_2 - 1). \tag{38}$$

Secondly, we take the sum over the first coordinate. Again we get two telescopic sums. Note

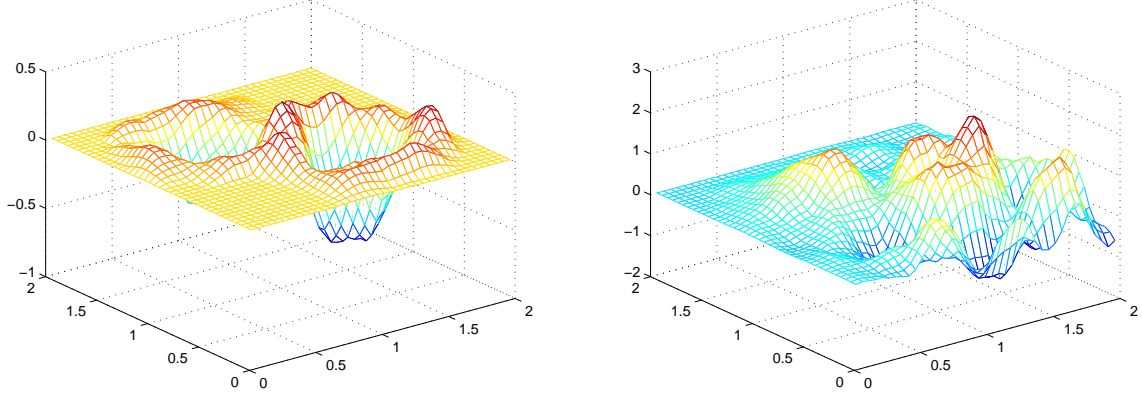


Figure 5: Left: the difference of the true density and the estimate. Right: f_{nh}^{+-} .

that $\lim_{i \rightarrow \infty} F(x_1 + i, x_2) = F_{Y_2}(x_2)$ and $\lim_{i \rightarrow \infty} F(x_1 + i, x_2 - 1) = F_{Y_2}(x_2 - 1)$, so we get

$$\begin{aligned}
& \sum_{i=1}^{\infty} g(x_1 + i, x_2) \\
&= \sum_{i=1}^{\infty} \{F(x_1 + i, x_2) - F(x_1 + i, x_2 - 1) - F(x_1 + i - 1, x_2) + F(x_1 + i - 1, x_2 - 1)\} \\
&= \sum_{i=1}^{\infty} \{F(x_1 + i, x_2) - F(x_1 + i - 1, x_2)\} + \sum_{i=1}^{\infty} \{F(x_1 + i - 1, x_2 - 1) - F(x_1 + i, x_2 - 1)\} \\
&= F_{Y_2}(x_2) - F(x_1, x_2) + F(x_1, x_2 - 1) - F_{Y_2}(x_2 - 1). \tag{39}
\end{aligned}$$

Thirdly, we sum over the second coordinate. Because $\lim_{j \rightarrow \infty} F_{Y_2}(x_2 - j) = 0$, this results in

$$\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} g(x_1 + i, x_2 - j) \\
&= \sum_{j=0}^{\infty} \{F_{Y_2}(x_2 - j) - F(x_1, x_2 - j) + F(x_1, x_2 - j - 1) - F_{Y_2}(x_2 - j - 1)\} \\
&= \sum_{j=0}^{\infty} \{F_{Y_2}(x_2 - j) - F_{Y_2}(x_2 - j - 1)\} - \sum_{j=0}^{\infty} \{F(x_1, x_2 - j) - F(x_1, x_2 - j - 1)\} \\
&= F_{Y_2}(x_2) - F(x_1, x_2) = F^{+-}(x_1, x_2). \tag{40}
\end{aligned}$$

Again, we can interchange the sums and we have shown (12). In similar fashion we can derive (11).

The last formula to recover $F(x_1, x_2)$ can be derived as follows. We begin with

$$g(x_1 + 1, x_2 + 1) = F(x_1, x_2) - F(x_1, x_2 + 1) - F(x_1 + 1, x_2) + F(x_1 + 1, x_2 + 1). \tag{41}$$

Now sum over the first coordinate to obtain

$$\sum_{i=1}^{\infty} g(x_1 + i, x_2 + 1) = F(x_1, x_2) - F(x_1, x_2 + 1) - F_{Y_2}(x_2) + F_{Y_2}(x_2 + 1). \quad (42)$$

Summing over the second coordinate we get

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} g(x_1 + i, x_2 + j) = F(x_1, x_2) - F_{Y_1}(x_1) - F_{Y_2}(x_2) + 1 = F^{++}(x_1, x_2). \quad (43)$$

Changing the order of summation again, we obtain (13).

The four inversion formulas for f are derived in a similar fashion. From (5) we have

$$\frac{\partial^2}{\partial x_1 \partial x_2} g(x_1, x_2) = f(x_1, x_2) - f(x_1, x_2 - 1) - f(x_1 - 1, x_2) + f(x_1 - 1, x_2 - 1).$$

Now, following equations (36) and (37), we obtain

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1 - i, x_2 - j) = f(x_1, x_2). \quad (44)$$

Here we have used $\lim_{x_1 \rightarrow -\infty} f(x_1, x_2) = 0$ and $\lim_{x_2 \rightarrow -\infty} f(x_1, x_2) = 0$.

The other three inversion formulas follow similarly. \square

6.2 Proof of Theorem 3.1

First we consider the estimator f_{nh}^{++} . We have

$$\begin{aligned} \mathbb{E} f_{nh}^{++}(x_1, x_2) &= \mathbb{E} \left(\frac{1}{nh^4} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w' \left(\frac{x_1 + i - X_{k1}}{h} \right) w' \left(\frac{x_2 + j - X_{k2}}{h} \right) \right) \\ &= \frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{E} w' \left(\frac{x_1 + i - X_{11}}{h} \right) w' \left(\frac{x_2 + j - X_{12}}{h} \right) \\ &= \frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w' \left(\frac{x_1 + i - u_1}{h} \right) w' \left(\frac{x_2 + j - u_2}{h} \right) g(u_1, u_2) du_1 du_2. \end{aligned} \quad (45)$$

Note that interchanging integrals and sums is allowed because

$$\frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| w' \left(\frac{x_1 + i - u_1}{h} \right) \right| \left| w' \left(\frac{x_2 + j - u_2}{h} \right) \right| g(u_1, u_2) du_1 du_2 < \infty. \quad (46)$$

To check this, we first make the substitutions $v_1 := u_1 - i$ and $v_2 := u_2 - j$. Secondly, we interchange the sums and integrals again, which is allowed because the integrand is nonnegative

(Fubini). We get

$$\begin{aligned} & \frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| w' \left(\frac{x_1 - v_1}{h} \right) \right| \left| w' \left(\frac{x_2 - v_2}{h} \right) \right| g(v_1 + i, v_2 + j) dv_1 dv_2 \\ &= \frac{1}{h^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| w' \left(\frac{x_1 - v_1}{h} \right) \right| \left| w' \left(\frac{x_2 - v_2}{h} \right) \right| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(v_1 + i, v_2 + j) dv_1 dv_2. \end{aligned} \quad (47)$$

Thirdly, noting that $F^{++}(v_1, v_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(v_1 + i, v_2 + j) dv_1 dv_2$ and that $F^{++}(v_1, v_2) \leq 1$, we obtain

$$\begin{aligned} & \frac{1}{h^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| w' \left(\frac{x_1 - v_1}{h} \right) \right| \left| w' \left(\frac{x_2 - v_2}{h} \right) \right| F^{++}(v_1, v_2) dv_1 dv_2 \\ & \leq \frac{1}{h^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| w' \left(\frac{x_1 - v_1}{h} \right) \right| \left| w' \left(\frac{x_2 - v_2}{h} \right) \right| dv_1 dv_2 < \infty. \end{aligned} \quad (48)$$

Because w' is a bounded function, and has bounded support, this integral is finite. Thus our use of Fubini's Theorem is justified. Next we apply partial integration twice, yielding

$$\begin{aligned} \mathbb{E} f_{nh}^{++}(x_1, x_2) &= \frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} w' \left(\frac{x_2 + j - u_2}{h} \right) \left(\int_{-\infty}^{\infty} w' \left(\frac{x_1 + i - u_1}{h} \right) g(u_1, u_2) du_1 \right) du_2 \\ &= -\frac{1}{h^3} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} w' \left(\frac{x_2 + j - u_2}{h} \right) \left(\int_{-\infty}^{\infty} w \left(\frac{x_1 + i - u_1}{h} \right) \frac{\partial}{\partial u_1} g(u_1, u_2) du_1 \right) du_2 \\ &= -\frac{1}{h^3} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} w \left(\frac{x_1 + i - u_1}{h} \right) \left(\int_{-\infty}^{\infty} w' \left(\frac{x_2 + j - u_2}{h} \right) \frac{\partial}{\partial u_1} g(u_1, u_2) du_2 \right) du_1 \\ &= \frac{1}{h^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w \left(\frac{x_1 + i - u_1}{h} \right) w \left(\frac{x_2 + j - u_2}{h} \right) \frac{\partial^2}{\partial u_1 \partial u_2} g(u_1, u_2) du_1 du_2. \end{aligned}$$

By the substitutions $v_1 := u_1 - i$ and $v_2 := u_2 - j$ we get

$$\begin{aligned} & \mathbb{E} f_{nh}^{++}(x_1, x_2) \\ &= \frac{1}{h^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w \left(\frac{x_1 - v_1}{h} \right) w \left(\frac{x_2 - v_2}{h} \right) \frac{\partial^2}{\partial u_1 \partial u_2} g(v_1 + i, v_2 + j) dv_1 dv_2. \end{aligned} \quad (49)$$

Now we need to interchange integrals and sums again. Therefore, rewrite the equation above as

$$\begin{aligned} & \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w \left(\frac{x_1 - v_1}{h} \right) w \left(\frac{x_2 - v_2}{h} \right) \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) dv_1 dv_2 \\ &= \lim_{M_1 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w \left(\frac{x_1 - v_1}{h} \right) w \left(\frac{x_2 - v_2}{h} \right) \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) dv_1 dv_2 \\ &= \lim_{M_1 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w \left(\frac{x_1 - v_1}{h} \right) w \left(\frac{x_2 - v_2}{h} \right) \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) dv_1 dv_2. \end{aligned} \quad (50)$$

In (38) we found that $g(v_1 + i, v_2) = F(v_1 + i, v_2) - F(v_1 + i, v_2 - 1) - F(v_1 + i - 1, v_2) + F(v_1 + i - 1, v_2 - 1)$, so

$$\frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2) = f(v_1 + i, v_2) - f(v_1 + i, v_2 - 1) - f(v_1 + i - 1, v_2) + f(v_1 + i - 1, v_2 - 1).$$

Following the summation of (39), we find

$$\begin{aligned} & \sum_{i=1}^{M_1} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2) \\ &= \sum_{i=1}^{M_1} \{f(v_1 + i, v_2) - f(v_1 + i, v_2 - 1) - f(v_1 + i - 1, v_2) + f(v_1 + i - 1, v_2 - 1)\} \\ &= \sum_{i=1}^{M_1} \{f(v_1 + i, v_2) - f(v_1 + i - 1, v_2)\} + \sum_{i=1}^{M_1} \{f(v_1 + i - 1, v_2 - 1) - f(v_1 + i, v_2 - 1)\} \\ &= f(v_1 + M_1, v_2) - f(v_1, v_2) - f(v_1, v_2 - 1) - f(v_1 + M_1, v_2 - 1) \end{aligned} \quad (51)$$

and

$$\begin{aligned} & \sum_{j=1}^{M_2} \sum_{i=1}^{M_1} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) \\ &= \sum_{j=1}^{M_2} \{f(v_1 + M_1, v_2 + j) - f(v_1, v_2 + j) - f(v_1, v_2 + j - 1) - f(v_1 + M_1, v_2 + j - 1)\} \\ &= \sum_{j=1}^{M_2} \{f(v_1 + M_1, v_2 + j) - f(v_1 + M_1, v_2 + j - 1) + \sum_{j=1}^{M_2} \{f(v_1, v_2 + j - 1) - f(v_1, v_2 + j)\}\} \\ &= f(v_1 + M_1, v_2 + M_2) - f(v_1 + M_1, v_2) + f(v_1, v_2) - f(v_1, v_2 + M_2). \end{aligned} \quad (52)$$

Note that this sum is finite for all v_1, v_2 , because f is bounded. Also note that changing the order of summation is allowed, because $M_1, M_2 < \infty$. Furthermore, in Theorem 2.1 we found that the sum converges to

$$\lim_{M_1 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) = f(v_1, v_2) < \infty. \quad (53)$$

We have assumed that f is bounded, so let $f(v_1, v_2) \leq \frac{1}{4}A$ for all v_1, v_2 , where $A > 0$ is a constant. Observe the following inequality

$$\begin{aligned} & |f(v_1 + M_1, v_2 + M_2) - f(v_1 + M_1, v_2) + f(v_1, v_2) - f(v_1, v_2 + M_2)| \\ & \leq |f(v_1 + M_1, v_2 + M_2)| + |f(v_1 + M_1, v_2)| + |f(v_1, v_2)| + |f(v_1, v_2 + M_2)| \\ & \leq A, \end{aligned} \quad (54)$$

for all v_1, v_2, M_1 , and M_2 . Note that, because w is nonnegative, bounded and has bounded support,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) dv_1 dv_2 < \infty \quad (55)$$

for all x_1, x_2 . Thus we can apply the Lebesgue Dominated Convergence Theorem to (49), and find

$$\begin{aligned} & \lim_{M_1 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) dv_1 dv_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) \lim_{M_1 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \frac{\partial^2}{\partial v_1 \partial v_2} g(v_1 + i, v_2 + j) dv_1 dv_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) f(v_1, v_2) dv_1 dv_2. \end{aligned} \quad (56)$$

Summarizing we now have

$$\mathbb{E} f_{nh}^{++}(x_1, x_2) = \frac{1}{h^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) f(v_1, v_2) dv_1 dv_2. \quad (57)$$

Substituting $z_1 := \frac{x_1 - v_1}{h}$ and $z_2 := \frac{x_2 - v_2}{h}$ we get

$$\mathbb{E} f_{nh}^{++}(x_1, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(z_1) w(z_2) f(x_1 - h z_1, x_2 - h z_2) dz_1 dz_2. \quad (58)$$

Using the multivariate version of Taylor's theorem derived in Wand and Jones (1995) for this particular application, allows us to rewrite

$$\begin{aligned} f(x_1 - h z_1, x_2 - h z_2) &= f(x_1, x_2) - h(z_1 f_1 + z_2 f_2)(x_1, x_2) \\ &\quad + \frac{1}{2} h^2 (z_1^2 f_{11} + z_1 z_2 (f_{12} + f_{21}) + z_2^2 f_{22})(x_1, x_2) + o(h^2). \end{aligned}$$

Using the symmetry of w , we obtain

$$\begin{aligned}
\mathbb{E} f_{nh}^{++}(x_1, x_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(z_1)w(z_2) \{f(x_1, x_2) - h(z_1 f_1 + z_2 f_2)(x_1, x_2) \\
&\quad + \frac{1}{2}h^2(z_1^2 f_{11} + z_1 z_2 (f_{12} + f_{21}) + z_2^2 f_{22})(x_1, x_2) + o(h^2)\} dz_1 dz_2 \\
&= f(x_1, x_2) - h f_1(x_1, x_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_1 w(z_1)w(z_2) dz_1 dz_2 \\
&\quad - h f_2(x_1, x_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_2 w(z_1)w(z_2) dz_1 dz_2 \\
&\quad + \frac{1}{2}h^2 f_{11}(x_1, x_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_1^2 w(z_1)w(z_2) dz_1 dz_2 \\
&\quad + \frac{1}{2}h^2 (f_{12} + f_{21})(x_1, x_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_1 z_2 w(z_1)w(z_2) dz_1 dz_2 \\
&\quad + \frac{1}{2}h^2 f_{22}(x_1, x_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_2^2 w(z_1)w(z_2) dz_1 dz_2 + o(h^2) \\
&= f(x_1, x_2) + \frac{1}{2}h^2 \int_{-\infty}^{\infty} z_1^2 w(z) dz (f_{11} + f_{22})(x_1, x_2) + o(h^2).
\end{aligned}$$

This proves statement (20) of the theorem for this individual estimator.

It is easily seen that

$$\begin{aligned}
\mathbb{E} f_{nh}^{--}(x_1, x_2) &= \mathbb{E} f_{nh}^{-+}(x_1, x_2) = \mathbb{E} f_{nh}^{+-}(x_1, x_2) = \mathbb{E} f_{nh}^{++}(x_1, x_2) \\
&= f(x_1, x_2) + \frac{1}{2}h^2 \int_{-\infty}^{\infty} z_1^2 w(z) dz (f_{11} + f_{22})(x_1, x_2) + o(h^2)
\end{aligned}$$

and thus $\mathbb{E} f_{nh}^{(t)}(x_1, x_2) = f(x_1, x_2) + \frac{1}{2}h^2 \int_{-\infty}^{\infty} z_1^2 w(z) dz (f_{11} + f_{22})(x_1, x_2) + o(h^2)$, proving equation (20).

Next let us derive the asymptotic variance. First, define

$$U_{kh}^{++}(x_1, x_2) := \frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w' \left(\frac{x_1 + i - X_{k1}}{h} \right) w' \left(\frac{x_2 + j - X_{k2}}{h} \right). \quad (59)$$

Then $f_{nh}^{++}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n U_{kh}^{++}(x_1, x_2)$, and since the terms U_{kh}^{++} are independent,

$$\text{Var}(f_{nh}^{++}(x_1, x_2)) = \frac{1}{n} \text{Var}(U_{1h}^{++}(x_1, x_2)). \quad (60)$$

Secondly, we will determine the variance of $U_{1h}^{++}(x_1, x_2)$. We have

$$\text{Var}(U_{1h}^{++}(x_1, x_2)) = \mathbb{E} U_{1h}^{++}(x_1, x_2)^2 - (\mathbb{E} U_{1h}^{++}(x_1, x_2))^2. \quad (61)$$

Let us begin with determining $\mathbb{E} U_{1h}^{++}(x_1, x_2)^2$. Note that, if $h < \frac{1}{2}$, we have

$$w' \left(\frac{x_1 + i_1 - X_{k1}}{h} \right) w' \left(\frac{x_2 + i_2 - X_{k2}}{h} \right) w' \left(\frac{x_1 + j_1 - X_{k1}}{h} \right) w' \left(\frac{x_1 + j_2 - X_{k2}}{h} \right) = 0 \quad (62)$$

unless $i_1 = i_2$ and $j_1 = j_2$, where $i_1, i_2, j_1, j_2 \in \mathbb{Z}$. This holds because if $i_1 \neq i_2$ or $j_1 \neq j_2$, then at least two pairs of arguments in the product (62) are more than distance two apart, rendering the product equal to zero. Thus in the following equation, as $h \rightarrow 0$, only the square products do not vanish and we can write

$$\begin{aligned} \mathbb{E} U_{1h}^{++}(x_1, x_2)^2 &= \mathbb{E} \left(\frac{1}{h^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w' \left(\frac{x_1 + i - X_{11}}{h} \right) w' \left(\frac{x_2 + j - X_{12}}{h} \right) \right)^2 \\ &= \frac{1}{h^8} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{E} \left(w' \left(\frac{x_1 + i - X_{11}}{h} \right) w' \left(\frac{x_2 + j - X_{12}}{h} \right) \right)^2. \end{aligned}$$

Now we use the substitutions $v_1 := u_1 - i$ and $v_2 := u_2 - j$ to obtain

$$\begin{aligned} \mathbb{E} U_{1h}^{++}(x_1, x_2)^2 &= \frac{1}{h^8} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(w' \left(\frac{x_1 + i - u_1}{h} \right) w' \left(\frac{x_2 + j - u_2}{h} \right) \right)^2 g(u_1, u_2) du_1 du_2 \\ &= \frac{1}{h^8} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(w' \left(\frac{x_1 - v_1}{h} \right) w' \left(\frac{x_2 - v_2}{h} \right) \right)^2 g(v_1 + i, v_2 + j) dv_1 dv_2. \end{aligned}$$

Note that the integrand is nonnegative, thus interchanging sums and integrals is allowed (Fubini), so

$$\begin{aligned} \mathbb{E} U_{1h}^{++}(x_1, x_2)^2 &= \frac{1}{h^8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(w' \left(\frac{x_1 - v_1}{h} \right) w' \left(\frac{x_2 - v_2}{h} \right) \right)^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(v_1 + i, v_2 + j) dv_1 dv_2 \\ &= \frac{1}{h^8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w' \left(\frac{x_1 - v_1}{h} \right)^2 w' \left(\frac{x_2 - v_2}{h} \right)^2 F^{++}(v_1, v_2) dv_1 dv_2. \end{aligned}$$

Now apply the substitutions $z_1 = (x_1 - v_1)/h$ and $z_2 = (x_2 - v_2)/h$ and recall the bounded support of w' . Furthermore, because $\lim_{h \rightarrow 0} F^{++}(x_1 - h z_1, x_2 - h z_2) = F^{++}(x_1, x_2) \leq 1$, we can again apply the Lebesgue Dominated Convergence Theorem

$$\begin{aligned} \mathbb{E} U_{1h}^{++}(x_1, x_2)^2 &= \frac{1}{h^6} \int_{-1}^1 \int_{-1}^1 w'(z_1)^2 w'(z_2)^2 F^{++}(x_1 - h z_1, x_2 - h z_2) dz_1 dz_2 \\ &= \frac{1}{h^6} \int_{-1}^1 \int_{-1}^1 w'(z_1)^2 w'(z_2)^2 F^{++}(x_1, x_2) dz_1 dz_2 + o(h^{-6}) \\ &= \frac{1}{h^6} F^{++}(x_1, x_2) \left(\int_{-1}^1 w'(z)^2 dz \right)^2 + o(h^{-6}). \end{aligned}$$

Now note that $\mathbb{E} U_{1h}^{++}(x_1, x_2) = \mathbb{E} f_{nh}^{++}(x_1, x_2) = f(x_1, x_2) + O(h^2)$. So

$$\begin{aligned} \text{Var}(f_{nh}^{++}(x_1, x_2)) &= \frac{1}{n} \text{Var}(U_{1h}^{++}(x_1, x_2)) \\ &= \frac{1}{n} \left[\mathbb{E} U_{1h}^{++}(x_1, x_2)^2 - (\mathbb{E} U_{1h}^{++}(x_1, x_2))^2 \right] \\ &= \frac{1}{n} \left[\frac{1}{h^6} F^{++}(x_1, x_2) \left(\int_{-1}^1 w'(z)^2 dz \right)^2 + o(h^{-6}) - f(x_1, x_2)^2 - O(h^2) \right] \\ &= \frac{1}{nh^6} F^{++}(x_1, x_2) \left(\int_{-1}^1 w'(z)^2 dz \right)^2 + o(n^{-1}h^{-6}). \end{aligned}$$

We can follow a similar procedure to obtain the variances of the other estimators. To summarize we get

$$\begin{aligned} \text{Var}(f_{nh}^{--}(x_1, x_2)) &= \frac{1}{nh^6} F^{--}(x_1, x_2) \left(\int_{-1}^1 w'(z)^2 dz \right)^2 + o(n^{-1}h^{-6}), \\ \text{Var}(f_{nh}^{-+}(x_1, x_2)) &= \frac{1}{nh^6} F^{-+}(x_1, x_2) \left(\int_{-1}^1 w'(z)^2 dz \right)^2 + o(n^{-1}h^{-6}), \\ \text{Var}(f_{nh}^{+-}(x_1, x_2)) &= \frac{1}{nh^6} F^{+-}(x_1, x_2) \left(\int_{-1}^1 w'(z)^2 dz \right)^2 + o(n^{-1}h^{-6}), \\ \text{Var}(f_{nh}^{++}(x_1, x_2)) &= \frac{1}{nh^6} F^{++}(x_1, x_2) \left(\int_{-1}^1 w'(z)^2 dz \right)^2 + o(n^{-1}h^{-6}). \end{aligned}$$

Now let us determine the variance of combinations of these estimators. We have

$$\begin{aligned} \text{Var}(f_{nh}^{(t)}(x_1, x_2)) &= \text{Var}(t_1 f_{nh}^{--}(x_1, x_2) + t_2 f_{nh}^{-+}(x_1, x_2) + t_3 f_{nh}^{+-}(x_1, x_2) + t_4 f_{nh}^{++}(x_1, x_2)) \\ &= t_1^2 \text{Var}(f_{nh}^{--}(x_1, x_2)) + t_2^2 \text{Var}(f_{nh}^{-+}(x_1, x_2)) + t_3^2 \text{Var}(f_{nh}^{+-}(x_1, x_2)) + t_4^2 \text{Var}(f_{nh}^{++}(x_1, x_2)) \\ &\quad + 2t_1 t_2 \text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{-+}(x_1, x_2)) + 2t_1 t_3 \text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{+-}(x_1, x_2)) \\ &\quad + 2t_1 t_4 \text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{++}(x_1, x_2)) + 2t_2 t_3 \text{Cov}(f_{nh}^{-+}(x_1, x_2), f_{nh}^{+-}(x_1, x_2)) \\ &\quad + 2t_2 t_4 \text{Cov}(f_{nh}^{-+}(x_1, x_2), f_{nh}^{++}(x_1, x_2)) + 2t_3 t_4 \text{Cov}(f_{nh}^{+-}(x_1, x_2), f_{nh}^{++}(x_1, x_2)). \end{aligned}$$

Let us look at $\text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{-+}(x_1, x_2))$. In similar fashion as we determined the variance, we find

$$\begin{aligned} \text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{-+}(x_1, x_2)) &= \frac{1}{n} \text{Cov}(U_{1h}^{--}(x_1, x_2), U_{1h}^{-+}(x_1, x_2)) \\ &= \frac{1}{n} [\mathbb{E} U_{1h}^{--}(x_1, x_2) U_{1h}^{-+}(x_1, x_2) - \mathbb{E} U_{1h}^{--}(x_1, x_2) \mathbb{E} U_{1h}^{-+}(x_1, x_2)] \end{aligned}$$

Let us first determine $\mathbb{E} U_{1h}^{--}(x_1, x_2) U_{1h}^{-+}(x_1, x_2)$. Note that, if $h < \frac{1}{2}$, we have

$$w' \left(\frac{x_1 - i_1 - X_{k1}}{h} \right) w' \left(\frac{x_2 - i_2 - X_{k2}}{h} \right) w' \left(\frac{x_1 - j_1 - X_{k1}}{h} \right) w' \left(\frac{x_1 + j_2 - X_{k2}}{h} \right) = 0, \quad (63)$$

for all i_1, i_2, j_1 and j_2 . This holds because the second and fourth argument in the product (63) are always more than distance two apart, rendering the product equal to zero. Thus

$$\mathbb{E} U_{1h}^{--}(x_1, x_2) U_{1h}^{++}(x_1, x_2) = 0. \quad (64)$$

Secondly, because we have already determined $\mathbb{E} U_{1h}^{--}(x_1, x_2)$ and $\mathbb{E} U_{1h}^{++}(x_1, x_2)$ earlier, we know that

$$\mathbb{E} U_{1h}^{--}(x_1, x_2) \mathbb{E} U_{1h}^{++}(x_1, x_2) = f(x_1, x_2)^2 + O(h^2). \quad (65)$$

Thus

$$\text{Cov}(f_{nh}^{--}(x_1, x_2), f_{nh}^{++}(x_1, x_2)) = \frac{1}{n}[-f(x_1, x_2)^2 - O(h^2)] = o(n^{-1}h^2). \quad (66)$$

This result holds for all the covariances. So we arrive at

$$\begin{aligned} \text{Var}(f_{nh}(x_1, x_2)) &= (t_1^2 F^{--}(x_1, x_2) + t_2^2 F^{++}(x_1, x_2) + t_3^2 F^{+-}(x_1, x_2) + t_4^2 F^{++}(x_1, x_2)) \\ &\quad \frac{1}{nh^6} \left(\int_{-1}^1 w'(z)^2 dz \right)^2 + o(n^{-1}h^{-6}) \\ &= B(x_1, x_2, t_1, t_2, t_3, t_4) \frac{1}{nh^6} \left(\int_{-1}^1 w'(z)^2 dz \right)^2 + o(n^{-1}h^{-6}). \end{aligned}$$

This proves statement (21) of the theorem. \square

6.3 Proof of Theorem 4.1

The convex combination of the four density estimators is given by

$$f_{nh}^{(t)}(x_1, x_2) = t_1 f_{nh}^{--}(x_1, x_2) + t_2 f_{nh}^{++}(x_1, x_2) + t_3 f_{nh}^{+-}(x_1, x_2) + t_4 f_{nh}^{++}(x_1, x_2), \quad (67)$$

where $t_1 + t_2 + t_3 + t_4 = 1$. Now define

$$\begin{aligned} S_{1nh}(x_1, x_2) &= f_{nh}^{--}(x_1, x_2) - f_{nh}^{+-}(x_1, x_2), \\ S_{2nh}(x_1, x_2) &= -f_{nh}^{++}(x_1, x_2) + f_{nh}^{+-}(x_1, x_2), \\ S_{3nh}(x_1, x_2) &= f_{nh}^{--}(x_1, x_2) - f_{nh}^{++}(x_1, x_2), \\ S_{4nh}(x_1, x_2) &= -f_{nh}^{+-}(x_1, x_2) + f_{nh}^{++}(x_1, x_2). \end{aligned}$$

We can rewrite (67) as

$$f_{nh}^{(t)}(x_1, x_2) = f_{nh}^{--}(x_1, x_2) - (t_3 + t_4) S_{1nh}(x_1, x_2) - t_2 S_{3nh}(x_1, x_2) + t_4 S_{4nh}(x_1, x_2), \quad (68)$$

Lemma 6.1 *Under the conditions of Theorem 4.1 we have, for $i = 1, \dots, 4$,*

$$\mathbb{E} S_{inh}(x_1, x_2) = 0, \quad (69)$$

$$\mathbb{E} S_{inh}(x_1, x_2)^2 = O\left(\frac{1}{nh^6}\right), \quad (70)$$

$$\mathbb{E} S_{inh}(x_1, x_2)^4 = O\left(\frac{1}{n^2 h^{12}}\right). \quad (71)$$

Proof

We give the proof for $S_{1nh}(x_1, x_2)$. The other claims can be proved similarly. Note that

$$\begin{aligned}
S_{1nh}(x_1, x_2) &= f_{nh}^{--}(x_1, x_2) - f_{nh}^{+-}(x_1, x_2) \\
&= \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w'\left(\frac{x_1 - i - X_{k1}}{h}\right) w'\left(\frac{x_2 - j - X_{k2}}{h}\right) \\
&\quad + \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} w'\left(\frac{x_1 + i - X_{k1}}{h}\right) w'\left(\frac{x_2 - j - X_{k2}}{h}\right) \\
&= \frac{1}{nh^4} \sum_{k=1}^n \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} w'\left(\frac{x_1 - i - X_{k1}}{h}\right) w'\left(\frac{x_2 - j - X_{k2}}{h}\right).
\end{aligned}$$

Define

$$U_{1kh}(x_1, x_2) := \frac{1}{h^4} \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} w'\left(\frac{x_1 + i - X_{k1}}{h}\right) w'\left(\frac{x_2 + j - X_{k2}}{h}\right). \quad (72)$$

Then $S_{1nh}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n U_{1kh}(x_1, x_2)$ and the terms in the sum are independent.

Following similar steps as in the proof of Theorem 3.1 we get

$$\begin{aligned}
\mathbb{E} U_{1kh}(x_1, x_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) \frac{\partial^2}{\partial v_1 \partial v_2} \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} g(v_1 + i, v_2 + j) dv_1 dv_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) \frac{\partial^2}{\partial v_1 \partial v_2} (1 - F_{Y_2}(v_2)) dv_1 dv_2 = 0.
\end{aligned}$$

We also have, as in the same proof,

$$\mathbb{E} S_{1nh}(x_1, x_2)^2 = \text{Var}(S_{1nh}(x_1, x_2)) = \frac{1}{n} \text{Var}(U_{11h}(x_1, x_2)) = O\left(\frac{1}{nh^6}\right).$$

Finally we consider the fourth moment of $S_{1nh}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n U_{1kh}(x_1, x_2)$. By independence of the terms we have

$$\begin{aligned}
\mathbb{E} S_{1nh}(x_1, x_2)^4 &= \frac{1}{n^3} \mathbb{E} U_{11h}(x_1, x_2)^4 + \frac{3(n-1)}{n^3} \left(\mathbb{E} U_{11h}(x_1, x_2)^2 \right)^2 \\
&= \frac{1}{n^3} O\left(\frac{1}{h^{14}}\right) + \frac{3(n-1)}{n^3} \left(O\left(\frac{1}{h^6}\right) \right)^2 = O\left(\frac{1}{n^2 h^{12}}\right).
\end{aligned}$$

This completes the proof of the lemma. \square

From (68) we get, omitting the arguments (x_1, x_2) ,

$$f_{nh}^{(\hat{t}_n)} - f_{nh}^{(\bar{t})} = -(\hat{t}_{n3} - \bar{t}_3)S_{1nh} - (\hat{t}_{n4} - \bar{t}_4)S_{1nh} - (\hat{t}_{n2} - \bar{t}_2)S_{3nh} + (\hat{t}_{n4} - \bar{t}_4)S_{4nh}. \quad (73)$$

Hence, under the assumptions of the theorem and by the Cauchy Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E} |f_{nh}^{(\hat{t}_n)} - f_{nh}^{(\bar{t})}| &\leq \mathbb{E} |\hat{t}_{n3} - \bar{t}_3| |S_{1nh}| + \mathbb{E} |\hat{t}_{n4} - \bar{t}_4| |S_{1nh}| + \mathbb{E} |\hat{t}_{n2} - \bar{t}_2| |S_{3nh}| + \mathbb{E} |\hat{t}_{n4} - \bar{t}_4| |S_{4nh}| \\
&\leq \left(\mathbb{E} (\hat{t}_{n3} - \bar{t}_3)^2 \right)^{1/2} \left(\mathbb{E} S_{1nh}^2 \right)^{1/2} + \left(\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^2 \right)^{1/2} \left(\mathbb{E} S_{1nh}^2 \right)^{1/2} \\
&\quad + \left(\mathbb{E} (\hat{t}_{n2} - \bar{t}_2)^2 \right)^{1/2} \left(\mathbb{E} S_{3nh}^2 \right)^{1/2} + \left(\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^2 \right)^{1/2} \left(\mathbb{E} S_{4nh}^2 \right)^{1/2} \\
&= \left(o(nh^{10}) O\left(\frac{1}{nh^6}\right) \right)^{1/2} = o(h^2).
\end{aligned}$$

Similarly we have, since $(y_1 + y_2 + y_3 + y_4)^2 \leq 4(y_1^2 + y_2^2 + y_3^2 + y_4^2)$,

$$\begin{aligned}
\text{Var}(f_{nh}^{(\hat{t}_n)} - f_{nh}^{(\bar{t})}) &\leq \mathbb{E} (f_{nh}^{(\hat{t}_n)} - f_{nh}^{(\bar{t})})^2 \\
&\leq 4\mathbb{E} (\hat{t}_{n3} - \bar{t}_3)^2 S_{1nh}^2 + 4\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^2 S_{1nh}^2 + 4\mathbb{E} (\hat{t}_{n2} - \bar{t}_2)^2 S_{3nh}^2 + 4\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^2 S_{4nh}^2 \\
&\leq 4 \left(\mathbb{E} (\hat{t}_{n3} - \bar{t}_3)^4 \right)^{1/2} \left(\mathbb{E} S_{1nh}^4 \right)^{1/2} + 4 \left(\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^4 \right)^{1/2} \left(\mathbb{E} S_{1nh}^4 \right)^{1/2} \\
&\quad + 4 \left(\mathbb{E} (\hat{t}_{n2} - \bar{t}_2)^4 \right)^{1/2} \left(\mathbb{E} S_{3nh}^4 \right)^{1/2} + 4 \left(\mathbb{E} (\hat{t}_{n4} - \bar{t}_4)^4 \right)^{1/2} \left(\mathbb{E} S_{4nh}^4 \right)^{1/2} \\
&= o(1) \left(O\left(\frac{1}{n^2 h^{12}}\right) \right)^{1/2} = o\left(\frac{1}{nh^6}\right).
\end{aligned}$$

Since the two bounds above are negligible compared to the order of the bias and variance in Theorem 3.1 it follows that this theorem also holds for the estimator with estimated weights.

In order to prove asymptotic normality note that by Lemma 6.1 and condition (32) it follows that $\sqrt{nh^3}$ times each of the terms in the representation (73) vanish in probability. Also it follows that $\sqrt{nh^3}$ times the expectation of (73) vanishes asymptotically. Hence the limit distributions of $\sqrt{nh^3}(f_{nh}^{(\hat{t}_n)} - \mathbb{E} f_{nh}^{(\hat{t}_n)})$ and $\sqrt{nh^3}(f_{nh}^{(\bar{t})} - \mathbb{E} f_{nh}^{(\bar{t})})$ coincide. The limit distribution of the latter follows by checking the Lyapounov condition for asymptotic normality. \square

6.4 Proof of Theorem 4.2

First we will expand the expected value for F_{nh}^{--} . We will skip the proofs for the remaining three two-dimensional estimators, since these can be done in precisely the same manner. We have

$$\begin{aligned}
\mathbb{E} F_{nh}^{--}(x_1, x_2) &= \\
&= \mathbb{E} \left(\frac{1}{nh^2} \sum_{k=1}^n \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w\left(\frac{x_1 - i - X_{k1}}{h}\right) w\left(\frac{x_2 - j - X_{k2}}{h}\right) \right) = \\
&= \frac{1}{h^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbb{E} w\left(\frac{x_1 - i - X_{11}}{h}\right) w\left(\frac{x_2 - j - X_{12}}{h}\right) = \\
&= \frac{1}{h^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - i - u_1}{h}\right) w\left(\frac{x_2 - j - u_2}{h}\right) g(u_1, u_2) du_1 du_2. \quad (74)
\end{aligned}$$

By substituting $v_1 := u_1 + i$ and $v_2 := u_2 + j$ and interchanging of integrals and sums we get

$$\begin{aligned} \mathbb{E} F_{nh}^{--}(x_1, x_2) &= \\ &= \frac{1}{h^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) g(v_1 - i, v_2 - j) dv_1 dv_2 = \\ &= \frac{1}{h^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g(v_1 - i, v_2 - j) dv_1 dv_2. \end{aligned} \quad (75)$$

Interchanging integrals and sums is allowed because the integrand is a nonnegative bounded function.

Further, since $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g(v_1 - i, v_2 - j) = F^{--}(v_1, v_2)$, we can continue with

$$\mathbb{E} F_{nh}^{--}(x_1, x_2) = \frac{1}{h^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) F^{--}(v_1, v_2) dv_1 dv_2. \quad (76)$$

Next we apply the substitutions $z_1 := (x_1 - v_1)/h$ and $z_2 := (x_2 - v_2)/h$ to get

$$\mathbb{E} F_{nh}^{--}(x_1, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(z_1) w(z_2) F^{--}(x_1 - h z_1, x_2 - h z_2) dz_1 dz_2. \quad (77)$$

The multivariate version of Taylor's theorem allows us to expand $F^{--}(x_1 - h z_1, x_2 - h z_2)$ as

$$\begin{aligned} F^{--}(x_1 - h z_1, x_2 - h z_2) &= F^{--}(x_1, x_2) - h(z_1 F_1^{--} + z_2 F_2^{--})(x_1, x_2) + \\ &+ \frac{1}{2} h^2 (z_1^2 F_{11}^{--} + z_1 z_2 (F_{12}^{--} + F_{21}^{--}) + z_2^2 F_{22}^{--})(x_1, x_2) + o(h^2), \end{aligned} \quad (78)$$

where $F_{11}^{--} = \frac{\partial^2 F^{--}(x_1, x_2)}{\partial x_1^2}$ and $F_{12}^{--} = \frac{\partial^2 F^{--}(x_1, x_2)}{\partial x_1 \partial x_2}$, etc.. Let us plug-in (78) into (77) and recall the function w is symmetric. Thus

$$\begin{aligned} \mathbb{E} F_{nh}^{--}(x_1, x_2) &= \int_{-1}^1 \int_{-1}^1 w(z_1) w(z_2) \{ F^{--}(x_1, x_2) - h(z_1 F_1^{--} + z_2 F_2^{--})(x_1, x_2) + \\ &+ \frac{1}{2} h^2 (z_1^2 F_{11}^{--} + z_1 z_2 (F_{12}^{--} + F_{21}^{--}) + z_2^2 F_{22}^{--})(x_1, x_2) + o(h^2) \} dz_1 dz_2 = \\ &= F^{--}(x_1, x_2) - h F_1^{--}(x_1, x_2) \int_{-1}^1 \int_{-1}^1 z_1 w(z_1) w(z_2) dz_1 dz_2 - \\ &- h F_2^{--}(x_1, x_2) \int_{-1}^1 \int_{-1}^1 z_2 w(z_1) w(z_2) dz_1 dz_2 + \\ &+ \frac{1}{2} h^2 F_{11}^{--}(x_1, x_2) \int_{-1}^1 \int_{-1}^1 z_1^2 w(z_1) w(z_2) dz_1 dz_2 + \\ &+ \frac{1}{2} h^2 (F_{12}^{--} + F_{21}^{--})(x_1, x_2) \int_{-1}^1 \int_{-1}^1 z_1 z_2 w(z_1) w(z_2) dz_1 dz_2 + \\ &+ \frac{1}{2} h^2 F_{22}^{--}(x_1, x_2) \int_{-1}^1 \int_{-1}^1 z_2^2 w(z_1) w(z_2) dz_1 dz_2 + o(h^2) = \\ &= F^{--}(x_1, x_2) + \frac{1}{2} h^2 (F_{11}^{--} + F_{22}^{--})(x_1, x_2) \int_{-1}^1 z^2 w(z) dz + o(h^2). \end{aligned} \quad (79)$$

Because w is supported only on $[-1, 1] \times [-1, 1]$ it is not necessary to integrate over all \mathbb{R} and we can change the domain of integration.

By following the same arguments for the other three estimators we obtain similar expansions for the expected values.

Let us continue with the proof of the variance expansion. Define

$$U_{kh}^{--}(x_1, x_2) := \frac{1}{h^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w\left(\frac{x_1 - i - X_{k1}}{h}\right) w\left(\frac{x_2 - j - X_{k2}}{h}\right). \quad (80)$$

Then $F_{nh}^{--}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n U_{kh}^{--}(x_1, x_2)$. Since all U_{kh}^{--} are independent, we have

$$\begin{aligned} \text{Var}(F_{nh}^{--}(x_1, x_2)) &= \frac{1}{n} \text{Var}(U_{1h}^{--}(x_1, x_2)) = \\ &= \frac{1}{n} \left(\mathbb{E}(U_{1h}^{--}(x_1, x_2))^2 - (\mathbb{E} U_{1h}^{--}(x_1, x_2))^2 \right). \end{aligned} \quad (81)$$

First we determine $\mathbb{E}(U_{1h}^{--}(x_1, x_2))^2$. Note that, if $h < \frac{1}{2}$, we have

$$w\left(\frac{x_1 - i_1 - X_{k1}}{h}\right) w\left(\frac{x_2 - i_2 - X_{k2}}{h}\right) w\left(\frac{x_1 - j_1 - X_{k1}}{h}\right) w\left(\frac{x_1 - j_2 - X_{k2}}{h}\right) = 0 \quad (82)$$

unless $i_1 = i_2$ and $j_1 = j_2$, where $i_1, i_2, j_1, j_2 \in \mathbb{Z}$. This holds because for any $i_1 \neq i_2$ or $j_1 \neq j_2$, at least one argument of w falls out of support rendering the product equal to zero. Thus in the following equation, as $h \rightarrow 0$, only the square products are not equal to zero and we can write

$$\begin{aligned} \mathbb{E}(U_{1h}^{--}(x_1, x_2))^2 &= \mathbb{E}\left(\frac{1}{h^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w\left(\frac{x_1 - i - X_{11}}{h}\right) w\left(\frac{x_2 - j - X_{12}}{h}\right)\right)^2 = \\ &= \frac{1}{h^4} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbb{E}\left(w\left(\frac{x_1 - i - X_{11}}{h}\right) w\left(\frac{x_2 - j - X_{12}}{h}\right)\right)^2. \end{aligned} \quad (83)$$

By substituting $v_1 := u_1 + i$ and $v_2 := u_2 + j$ and interchanging of integrals and sums, which is allowed because integrand is nonnegative, we get

$$\begin{aligned} \mathbb{E}(U_{1h}^{--}(x_1, x_2))^2 &= \\ &= \frac{1}{h^4} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(w\left(\frac{x_1 - i - u_1}{h}\right) w\left(\frac{x_2 - j - u_2}{h}\right) \right)^2 g(u_1, u_2) du_1 du_2 = \\ &= \frac{1}{h^4} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) \right)^2 g(v_1 - i, v_2 - j) dv_1 dv_2 = \\ &= \frac{1}{h^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(w\left(\frac{x_1 - v_1}{h}\right) w\left(\frac{x_2 - v_2}{h}\right) \right)^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g(v_1 - i, v_2 - j) dv_1 dv_2 = \\ &= \frac{1}{h^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^2\left(\frac{x_1 - v_1}{h}\right) w^2\left(\frac{x_2 - v_2}{h}\right) F^{--}(v_1, v_2) dv_1 dv_2. \end{aligned}$$

Next we apply the substitutions $z_1 := (x_1 - v_1)/h$ and $z_2 := (x_2 - v_2)/h$. The fact that $\lim_{h \rightarrow 0} F^{--}(x_1 - hz_1, x_2 - hz_2) = F^{--}(x_1, x_2) \leq 1$ then yields by the dominated convergence theorem

$$\begin{aligned} \mathbb{E}(U_{1h}^{--}(x_1, x_2))^2 &= \frac{1}{h^2} \int_{-1}^1 \int_{-1}^1 w^2(z_1)w^2(z_2)F^{--}(x_1 - hz_1, x_2 - hz_2)dz_1dz_2 = \\ &= \frac{1}{h^2} \int_{-1}^1 \int_{-1}^1 w^2(z_1)w^2(z_2)F^{--}(x_1, x_2)dz_1dz_2 + o(h^{-2}) = \\ &= \frac{1}{h^2} F^{--}(x_1, x_2) \left(\int_{-1}^1 w^2(z)dz \right)^2 + o(h^{-2}). \end{aligned} \quad (84)$$

Because w has support only on $[-1, 1] \times [-1, 1]$ we are allowed to change the domain of integration.

For the term $(\mathbb{E} U_{1h}^{--}(x_1, x_2))^2$ note that

$$\mathbb{E} U_{1h}^{--}(x_1, x_2) = \mathbb{E} F_{nh}^{--}(x_1, x_2) = F^{--}(x_1, x_2) + O(h^2) \quad (85)$$

So the variance of $F_{nh}^{--}(x_1, x_2)$ is given by

$$\begin{aligned} \text{Var}(F_{nh}^{--}(x_1, x_2)) &= \\ &= \frac{1}{n} \left[\frac{1}{h^2} F^{--}(x_1, x_2) \left(\int_{-1}^1 w^2(z)dz \right)^2 + o(h^{-2}) - (F^{--}(x_1, x_2) + O(h^2))^2 \right] = \end{aligned} \quad (86)$$

$$= \frac{1}{nh^2} F^{--}(x_1, x_2) \left(\int_{-1}^1 w^2(z)dz \right)^2 + o(n^{-1}h^{-2}). \quad (87)$$

Likewise we may determine the other variances of the two-dimensional distribution estimators. \square

6.5 Proof of lemma 4.3

Proof Let us first introduce some notation. Define the vectors $\mathbf{v}(x_1, x_2)$ and $\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2)$ by

$$\begin{aligned} \mathbf{v}(x_1, x_2) &= (F^{--}(x_1, x_2), F^{-+}(x_1, x_2), F^{+-}(x_1, x_2), F^{++}(x_1, x_2)), \\ \tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2) &= (\tilde{F}_{n\tilde{h}}^{--}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{-+}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{+-}(x_1, x_2), \tilde{F}_{n\tilde{h}}^{++}(x_1, x_2)). \end{aligned}$$

Note that, for n large enough, the components of these vectors are all at least ϵ_n and that they are at most one.

We will only check (27) and (29) for i equal to one. The other cases can be treated similarly. Then we also need the vector of partial derivatives of the the function $\tilde{t}_1(y_1, y_2, y_3, y_4)$. Note that on the line segment between $\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2)$ and $\mathbf{v}(x_1, x_2)$ all the components are all at least ϵ_n and that they are at most one. This implies after some computation

$$\|\nabla \tilde{t}_1(y_1, y_2, y_3, y_4)\|^2 \leq \frac{B}{\epsilon_n^6},$$

for some constant B , for all points (y_1, y_2, y_3, y_4) on this line segment.

We can now apply the multivariate mean value theorem and the Cauchy Schwarz inequality to get

$$\begin{aligned} (\hat{t}_{n1}(x_1, x_2) - \bar{t}_1(x_1, x_2))^2 &= (\tilde{t}_1(\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2)) - \tilde{t}_1(\mathbf{v}(x_1, x_2)))^2 \\ &= (\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2) - \mathbf{v}(x_1, x_2)) \cdot \nabla \tilde{t}_1(y_1, y_2, y_3, y_4)^2 \\ &\leq \|\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2) - \mathbf{v}(x_1, x_2)\|^2 \|\nabla \tilde{t}_1(y_1, y_2, y_3, y_4)\|^2 \\ &\leq \frac{B}{\epsilon_n^6} \|\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2) - \mathbf{v}(x_1, x_2)\|^2, \end{aligned}$$

where (y_1, y_2, y_3, y_4) is a point on the line segment between $\tilde{\mathbf{v}}(x_1, x_2)_{n\tilde{h}}$ and $\mathbf{v}(x_1, x_2)$. Note that $\|\tilde{\mathbf{v}}_{n\tilde{h}}(x_1, x_2) - \mathbf{v}(x_1, x_2)\|^2$ is a sum of four terms like $(\tilde{F}_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^2$, which is smaller than $(F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^2$, and that $E(F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^2$ equals the variance plus the squared bias of $F_{n\tilde{h}}^{--}(x_1, x_2)$. By Theorem 4.2 we can bound these to get

$$E(\hat{t}_{n1}(x_1, x_2) - \bar{t}_1(x_1, x_2))^2 \leq \frac{B}{\epsilon_n^6} \left(O\left(\frac{1}{n\tilde{h}^2}\right) + O(\tilde{h}^4) \right) = O(n^{-2/3}(\log n)^6),$$

for a bandwidth h of order $n^{-1/6}$. This implies that (27) is satisfied.

Let us now check that (29) is satisfied. By an argument similar to the one above it suffices to check if terms like $E(F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^4$ vanish asymptotically. Write

$$F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2) = F_{n\tilde{h}}^{--}(x_1, x_2) - E F_{n\tilde{h}}^{--}(x_1, x_2) + E F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2).$$

By the triangle inequality we have

$$\begin{aligned} \left(E(F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^4 \right)^{1/4} &\leq \\ &\leq \left(E(F_{n\tilde{h}}^{--}(x_1, x_2) - E F_{n\tilde{h}}^{--}(x_1, x_2))^4 \right)^{1/4} + \left(E F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2) \right)^{1/4}. \end{aligned}$$

So, by $(a + b)^4 \leq 8(a^4 + b^4)$, $a, b \geq 0$, we also have

$$\begin{aligned} E(F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2))^4 &\leq \\ &\leq 8E(F_{n\tilde{h}}^{--}(x_1, x_2) - E F_{n\tilde{h}}^{--}(x_1, x_2))^4 + 8\left(E F_{n\tilde{h}}^{--}(x_1, x_2) - F^{--}(x_1, x_2)\right)^4. \end{aligned}$$

Since the bias vanishes by Theorem 4.2 it suffices to prove the bound of the lemma for the fourth power of the error.

Recall from the proof of Theorem 4.2 that $F_{n\tilde{h}}^{--}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n U_{k\tilde{h}}^{--}(x_1, x_2)$, where

$$U_{k\tilde{h}}^{--}(x_1, x_2) := \frac{1}{\tilde{h}^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w\left(\frac{x_1 - i - X_{k1}}{\tilde{h}}\right) w\left(\frac{x_2 - j - X_{k2}}{\tilde{h}}\right).$$

Note that the $U_{k\tilde{h}}^{--}$ are independent. Now write

$$F_{n\tilde{h}}^{--}(x_1, x_2) - E F_{n\tilde{h}}^{--}(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n \tilde{U}_{k\tilde{h}}^{--}(x_1, x_2),$$

where $\tilde{U}_{k\tilde{h}}^{--}(x_1, x_2) = U_{k\tilde{h}}^{--}(x_1, x_2) - \mathbb{E} U_{k\tilde{h}}^{--}(x_1, x_2)$. Since $\mathbb{E} \tilde{U}_{k\tilde{h}}^{--}(x_1, x_2)$ equals zero we have

$$\mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n \tilde{U}_{k\tilde{h}}^{--}(x_1, x_2) \right)^4 = \frac{1}{n^3} \mathbb{E} \left(\tilde{U}_{k\tilde{h}}^{--}(x_1, x_2)^4 \right) + 3 \frac{n-1}{n^3} \left(\mathbb{E} \left(\tilde{U}_{k\tilde{h}}^{--}(x_1, x_2)^2 \right) \right)^2.$$

Similar to the derivation of (84) we get

$$\frac{1}{n^3} \mathbb{E} \left(\tilde{U}_{k\tilde{h}}^{--}(x_1, x_2)^4 \right) \sim \frac{1}{n^3} \mathbb{E} \left(U_{k\tilde{h}}^{--}(x_1, x_2)^2 \right) \sim O\left(\frac{1}{n^3 \tilde{h}^4}\right)$$

and

$$3 \frac{n-1}{n^3} \left(\mathbb{E} \left(\tilde{U}_{k\tilde{h}}^{--}(x_1, x_2)^2 \right) \right)^2 = 3 \frac{n-1}{n^3} \left(\text{Var}(U_{k\tilde{h}}^{--}(x_1, x_2)) \right)^2 \sim O\left(\frac{1}{n^2 \tilde{h}^4}\right).$$

Under the condition on \tilde{h} in the lemma both terms vanish. This shows that (29) is satisfied as well. Condition (32) follows from condition (29) by the Cauchy-Schwarz inequality. \square

6.6 An inequality

The next lemma can be used to derive the weights that minimize the asymptotic variance of the convex combination of the original for estimators of the density f .

Lemma 6.2 *Let a_1, \dots, a_m be m positive numbers. Then for all positive t_1, \dots, t_m with $t_1 + \dots + t_m = 1$ we have*

$$a_1 t_1^2 + \dots + a_m t_m^2 \geq \frac{a_1 a_2 \dots a_m}{s_m(a_1, \dots, a_m)}, \quad (88)$$

where $s_m(a_1, \dots, a_m)$ is defined by

$$s_m(a_1, \dots, a_m) = a_2 a_3 \dots a_m + \sum_{j=2}^{m-1} a_1 \dots a_{j-1} a_{j+1} \dots a_m + a_1 a_2 \dots a_{m-1}, \quad (89)$$

the sum of the m products of length $m-1$ obtained by skipping one term in the full product.

The minimum is attained at the t vector given by $t_1 = a_2 a_3 \dots a_m / s_m(a_1, \dots, a_m)$ and $t_m = a_1 a_2 \dots a_{m-1} / s_m(a_1, \dots, a_m)$ and

$$t_i = \frac{a_1 a_2 \dots a_{i-1} a_{i+1} \dots a_m}{s_m(a_1, \dots, a_m)}, \quad i = 2, \dots, m-1.$$

Proof Introduce the inner product $\langle \cdot, \cdot \rangle_a$ and corresponding norm $\|\cdot\|_a$ by

$$\langle x, y \rangle_a = a_2 a_3 \dots a_m x_1 y_1 + a_1 a_3 \dots a_m x_2 y_2 + \dots + a_1 a_2 \dots a_{m-1} x_m y_m, \quad (90)$$

$$\|x\|_a = \left(a_2 a_3 \dots a_m x_1^2 + a_1 a_3 \dots a_m x_2^2 + \dots + a_1 a_2 \dots a_{m-1} x_m^2 \right)^{1/2}. \quad (91)$$

Then, with $\mathbf{1}$ equal to the vector of m ones, the Cauchy-Schwarz inequality implies

$$\begin{aligned}
a_1 a_2 \dots a_m &= (a_1 a_2 \dots a_m)(t_1 + t_2 + \dots + t_m) \\
&= \langle \mathbf{1}, (a_1 t_1, a_2 t_2, \dots, a_m t_m) \rangle_a \leq \| \mathbf{1} \|_a \| (a_1 t_1, a_2 t_2, \dots, a_m t_m) \|_a \\
&= \sqrt{s(a_1, \dots, a_m)} \left(a_2 a_3 \dots a_m (a_1 t_1)^2 + a_1 a_3 \dots a_m (a_2 t_2)^2 + \dots + a_1 a_2 \dots a_{m-1} (a_m t_m)^2 \right)^{1/2} \\
&= \sqrt{s(a_1, \dots, a_m)} \left((a_1 a_2 \dots a_m) (a_1 t_1^2 + a_2 t_2^2 + \dots + a_m t_m^2) \right)^{1/2},
\end{aligned}$$

which implies the inequality after some rewriting. \square

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